STAT 462/662
Introduction to Stochastic Processes

Homework 3

Due: Tuesday September 21, 2021

1. Successively flip a (fair) quarter until the pattern HHHT appears, that is, until you observe three successive heads followed by a tails. In order to calculate some properties of this game, define a Markov chain with the following states: 0, H, HH, HHH, and HHHT, where 0 represents the starting point, H represents a single observed head on the last flip, HH represents two successive heads on the last two flips, HHH represents three successive heads on the last three flips, and HHHT is the sequence that you are looking for. Find the transition probability matrix.

   Hint: For example, if we observed THH, the process is in state HH. If we observe a head next time, the pattern is THHH and the process makes a transition from state HH to state HHH.

2. **(Stat 662 only; Bonus for 462)** Two urns A and B contain a total of N balls. Assume that at time n, there are exactly i balls in A.

   (Step 1) At time $n+1$, an urn is selected at random in proportion to its contents (i.e., A is chosen with probability i/N and B is chosen with probability $(N-i)/N$).

   (Step 2) Then, a ball is selected from A with probability p or from B with probability q and placed in the urn chosen in Step 1.

   Now let's assume $N = 4$. Determine the transition matrix probability for this Markov chain.

3. A Markov chain $\{X_n : \ n = 0, 1, 2, \ldots\}$ has the transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.2 & 0.4 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{pmatrix}.$$

   If it is known that the process starts in state 0, determine the probability $P(X_3 = 2)$.

4. Consider the problem of sending a binary message, 0 or 1, through a signal channel consisting of several stages, where transmission through each stage is subject to a fixed probability of error 0.2. Let X_0 be the signal that is sent, and let X_n be the signal that is received at the nth stage. Suppose that X_n is a Markov chain with transition probabilities $P_{00} = P_{11} = 0.8$ and $P_{01} = P_{10} = 0.2$. Determine $P(X_{10} = 0|X_0 = 0)$, the probability of correct transmission through ten stages.

5. Suppose that coin 1 has probability 0.7 of coming up heads, and coin 2 has probability 0.6 of coming up heads. If the coin flipped today comes up heads, then we select coin 1 to flip tomorrow. If it comes up tails, then we select coin 2 to flip tomorrow.

(a) Define X_n and state space.

(b) Find a transition probability matrix.

(c) Suppose that the coin flipped on Monday comes up heads. What is the probability that the coin flipped on Friday of the same week also comes up heads?

6. A Markov chain on states $\{0, 1, 2, 3, 4, 5\}$ has transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/3 & 0 & 2/3 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 3/4 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 1/5 & 0 & 4/5 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix}.$$

Find all communicating classes; which classes are transient and which are recurrent?

7. A Markov chain on states $\{0, 1, 2, 3, 4, 5\}$ has transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3/4 & 1/4 & 0 & 0 & 0 \\ 0 & 1/8 & 7/8 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/8 & 3/8 & 0 \\ 1/3 & 0 & 1/6 & 1/4 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Find all communicating classes; which classes are transient and which are recurrent?

8. A Markov chain on states $\{0, 1, 2, 3, 4, 5\}$ has transition probability matrix

$$\begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1/3 & 1/3 & 0 & 1/3 \end{pmatrix}.$$

Find all communicating classes; which classes are transient and which are recurrent?