Lecture 5 Stat 462/662 9/7/21

§4.1 continued (§3.3 in Pinsky & Karlin)

More examples of Markov chains – discrete in time & space

Example 1: Ehrenfest urn model

* A classical mathematical description of diffusion through a membrane is the well-known Ehrenfest urn model.

2 containers (think of 2 boxes or "urns"):

- contain a total of $2a$ balls (molecules)
- A ball is selected at random (all selections equally likely) from the $2a$ balls & moved to the other container → this models 1 molecule diffusing randomly through the membrane
- Each selection generates a transition of the process

① Let Y_n = # of balls in urn A at time n (n^{th} stage)

$S_Y = \{0, 1, \ldots, 2a\}$

Then $\{Y_n : n = 0, 1, \ldots\}$ is a MC.

Transition Probabilities : $P_{ij} = P(Y_{n+1} = j \mid Y_n = i)$

$$P_{ij} = \begin{cases} \frac{2a-i}{2a} & \text{if } i \neq 0, 2a \text{ and } j = i+1 \\ \frac{i}{2a} & \text{if } i \neq 0, 2a \text{ and } j = i-1 \\ 1 & \text{if } i = 0 \text{ and } j = 1 \\ 1 & \text{if } i = 2a \text{ and } j = 2a-1 \\ 0 & \text{otherwise} \end{cases}$$

more intuitive

② Define $X_n = \frac{Y_n - (2a - Y_n)}{2} = Y_n - a$

difference b/t the 2 urns divided by 2

$S_X = \{-a, \dots, 0, \dots, a\}$

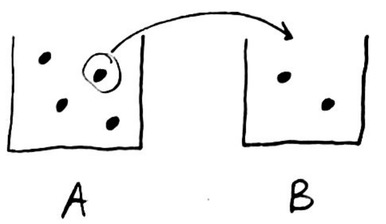
Then $\{X_n : n = 0, 1, \dots\}$ is a MC.

Transition Probabilities: $P(X_{n+1} = j \mid X_n = i)$

$$P_{ij} = \begin{cases} \frac{a-i}{2a} & \text{if } i \neq -a, a \text{ and } j = i+1 \\ \frac{a+i}{2a} & \text{if } i \neq -a, a \text{ and } j = i-1 \\ 1 & \text{if } i = -a \text{ and } j = -a+1 \\ 1 & \text{if } i = a \text{ and } j = a-1 \\ 0 & \text{otherwise} \end{cases}$$

more commonly used version

In version ① Say $a = 3$, so total # balls $= 2(3) = 6$, $S_Y = \{0, \dots, 6\}$



$$P_{43} = P(Y_{n+1} = 3 \mid Y_n = 4) = \frac{i}{2a} = \frac{4}{6}$$

Urn A : $4 \to 3$

i $\quad$ $j = i-1$

In version (2)

$$X_n = Y_n - a = 4 - 3 = 1 \quad , \quad S_X = \{\overset{-a}{-3}, \dots, \overset{a}{3}\}$$

$$P_{10} = \frac{a+i}{2a} = \frac{3+1}{6} = \frac{4}{6}$$

$i=1$ $j=i-1=0$

Note: If X_n reaches $-a$ or a, it goes back to states $-a+1$ or $a-1$ (resp.) with probability 1.

def: The 2 barriers $-a$ & a in this example are reflective (bounce back w/ prob. 1).

* The urn with smaller # of balls has a greater chance of getting a ball than losing one

→ just like diffusion: flow from higher to lower density

Example 2: Queue

Customers arrive for a service & wait in line until their turn.

Time is discrete: time to serve 1 customer is 1 unit (fixed).

(If no customer is waiting, then no service is performed
→ e.g. Taxi stand, cab arrives at fixed intervals

and if no one is there waiting, cab immediately departs).

Let ξ_n = # of new customers arriving at time n.

Assume $\xi_1, \xi_2, \dots$ are i.i.d. RVs

$$P(\xi_n = k) = \begin{cases} a_k & \text{if } k = 0, 1, \dots \\ 0 & \text{if } k < 0 \end{cases}$$

where $a_k \geq 0$ and $\sum_{k=0}^{\infty} a_k = 1$.

Define X_n = # of people waiting in line for service at time n.

If $X_n \geq 1$, then $X_{n+1} = X_n - \underbrace{1}_{\text{person being served}} + \xi_{n+1}$

If $X_n = 0$, then $X_{n+1} = \xi_{n+1}$

Then $\{X_n : n = 0, 1, \dots\}$ is a MC with

Transition probabilities $P_{ij} = P(X_{n+1} = j \mid X_n = i)$ s.t.

$$P_{ij} = \begin{cases} P(\xi_{n+1} = j - i + 1) & \text{if } i \geq 1 \\ P(\xi_{n+1} = j) & \text{if } i = 0 \end{cases}$$

$$= \begin{cases} a_{j-i+1} & \text{if } i \geq 1 \\ a_j & \text{if } i = 0 \end{cases} \quad \text{recall that } (a_k = 0 \text{ if } k < 0)$$

Transition probability matrix:

$$
P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \end{array} \\
\left[\begin{array}{ccccc}
a_0 & a_1 & a_2 & a_3 & \cdots \\
a_0 & a_1 & a_2 & a_3 & \\
0 & a_0 & a_1 & a_2 & \\
0 & 0 & a_0 & a_1 & \\
\vdots & & & & \ddots
\end{array}\right]
\end{array}
$$

→ all a_j's since $i=0$

→ all a_j's also, since $i=1$

→ $i=2$ so $a_{j-i+1} = a_{j-2+1}$ ↗ 0 for $j=0$ ↘ a_{j-1} for $j \geq 1$

e.g. $P_{24} = P(X_{n+1} = 4 \mid X_n = 2) = P(\xi_{n+1} = 3)$

$= P(\xi_{n+1} = 4-2+1) = a_3$

time n		time $n+1$	
X	→ gets service next	X	
X		X	need to add 3 new customers for $X_{n+1} = 4$
		X	
		X	

Q. what will happen to the queue length as $n \to \infty$?

Ans: It depends on the mean number of arrivals

$$\mu = E[\xi] = \sum_{k=0}^{\infty} k \, a_k$$

- If $\mu < 1$, the queue becomes stable. (approaches a statistical equilibrium described by a limiting dist'n.)
- If $\mu \geq 1$, the queue explodes!

Example 3 : Success Runs

Consider a Markov chain on the nonneg. integers ($\mathbb{Z}^+$) with 2 possible outcomes : success (S) or failure (F), repeated trials (i.i.d.).

e.g. Dow Jones up or down for the day

$P(S) = p$ & $P(F) = q = 1-p$

Let X_n = run length of successes at time n
= # of successes since the last failure

$X_n = 0$ if the outcome (at time n) is F

Transition Probabilities: $P_{ij} = P(X_{n+1} = j \mid X_n = i)$

$$P_{ij} = \begin{cases} p & \text{if } j = i+1 \\ q & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases}$$

Markov since trials are indep.

$$\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{bmatrix} q & p & 0 & 0 & \cdots & \\ q & 0 & p & 0 & \cdots & \\ q & 0 & 0 & p & \cdots & \\ q & 0 & 0 & 0 & p & \cdots \\ \vdots & & & & \ddots & \end{bmatrix} \end{array}$$

<u>Notation</u>:

Success run of length r at trial n if outcomes in the preceeding $r+1$ trials including current trial were $F\underbrace{SS\cdots S}_{r \text{ successes}}$.