Lecture 6 | Stat 462/662 | 9/9/21 ①

§4.2 Chapman-Kolmogorov Equations

Recall: $P_{ij} = P(X_{n+1} = j \mid X_n = i)$ - 1-step transition probabilities

$\mathbb{P}$ - transition prob. matrix (1-step)

Q. What is the long-term behavior of the Markov chain?

Define the n-step transition probability:

$P_{ij}^{(n)}$ = probability that a process in state i will be in state j after n transitions

$$= P(X_{n+m} = j \mid X_m = i), \quad n \geq 0 \text{ \& } i,j \geq 0$$

For $n=1$,

$$P_{ij}^{(1)} = P(X_{m+1} = j \mid X_m = i) = P_{ij}$$

1-step transition prob.

For $n=0$,

$$P_{ij}^{(0)} = P(X_m = j \mid X_m = i) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Let $\mathbb{P}^{(n)}$ denote the matrix of n-step transition prob. $P_{ij}^{(n)}$.

def: Chapman-Kolmogorov equations – provide a method for computing the n-step transition probs.

For any $m, n \geq 0$ & states $i, j \in S$,

$$\boxed{P_{ij}^{(n+m)} = \sum_{k \in S} P_{ik}^{(n)} P_{kj}^{(m)}}$$

intermediate states

$\Rightarrow$ Starting in state i, the process will go to state j in $n+m$ transitions through a path which takes it into state k at the n^{th} transition.

$\rightarrow$ prob. of that is given by $P_{ik}^{n} P_{kj}^{m}$

Summing over all possible intermed. states k gives the prob. that the process will be in state j after $n+m$ transitions.

Proof: $P_{ij}^{(n+m)} = P(X_{n+m} = j \mid X_0 = i)$

$$= \sum_k P(X_{n+m} = j, X_n = k \mid X_0 = i)$$

$$= \sum_k \underbrace{P(X_{n+m} = j \mid X_n = k, X_0 = i)}_{P_{kj}^{(m)}} \underbrace{P(X_n = k \mid X_0 = i)}_{P_{ik}^{(n)}}$$

$$= \sum_k P_{ik}^{(n)} \cdot P_{kj}^{(m)}$$

Transition prob. matrix $\mathbb{P} = (P_{ij})_{i,j \in S}$

$$\mathbb{P}^{(2)} = \mathbb{P} \cdot \mathbb{P} = \left(\sum_{k \in S} P_{ik} P_{kj} \right) = \left(P_{ij}^{(2)} \right)$$

$\Rightarrow$ $P_{ij}^{(2)}$ is the $(i,j)^{\text{th}}$ entry of $\mathbb{P}^{(2)}$

Likewise, $P_{ij}^{(n)}$ is the $(i,j)^{\text{th}}$ entry of $\mathbb{P}^{(n)}$

Note : $P_{ij}^{(n)} \neq (P_{ij})^n$ ← careful with notation!

$\mathbb{P}^{(n)}$ is the matrix of n-step transition prob. $P_{ij}^{(n)}$

$\mathbb{P}^{(n+m)} = \mathbb{P}^{(n)} \cdot \mathbb{P}^{(m)}$ by Chapman-Kolmogorov

↗ matrix multiplication

$\mathbb{P}^{(2)} = \mathbb{P}^{(1+1)} = \mathbb{P} \cdot \mathbb{P} = \mathbb{P}^2$ & by induction we have

$\mathbb{P}^{(n)} = \mathbb{P}^{(n-1+1)} = \mathbb{P}^{(n-1)} \cdot \mathbb{P} = \mathbb{P}^n$

* $\boxed{\text{Thus, } \mathbb{P}^{(n)} \text{ is obtained by multiplying } \mathbb{P} \text{ by iself } n \text{ times!}}$

Example 1 : Back to Forecasting the weather

2 states : $\{0, 1\}$ — 0 = rain, 1 = no rain

discrete time $T = \{0, 1, \dots\}$

$$\mathbb{P} = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0.7 & 0.3 \\ 1 & 0.4 & 0.6 \end{array}$$

a) Calculate the probability that it will rain 2 days from today given that it is raining today.

Given that $X_n = 0$, don't know if X_{n+1} is 0 or 1, want prob. that $X_{n+2} = 0$.

$$P_{00}^{(2)} = \sum_{k=0}^{1} P_{0k} P_{k0} = P_{00} P_{00} + P_{01} P_{10}$$

$$= (0.7)(0.7) + (0.3)(0.4)$$

$$= 0.61$$

b) Calculate the prob. that it will rain 4 days from today given that it's raining today.

Want $P_{00}^{(4)}$. Compute $\mathbb{P}^{(4)}$ by multiplying $\mathbb{P}$ by itself 4 times :

$$P^4 = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{bmatrix}.$$

So, $P_{00}^{(4)} = 0.5749$

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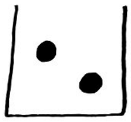
Example 2: Urn Problem

An urn always contains 2 balls (red or blue).



At each stage, a ball is randomly chosen & then replaced by a new ball:

- same color w/prob. 0.8
- opposite color w/prob. 0.2

Q. If initially both balls are red, what is the prob. that the 5^{th} ball selected is red?

First define a MC.

Note: Probability that a selection is red is determined by the composition of urn at time of selection.

Let X_n = # of red balls in urn after the n^{th} selection (& replacement).

$S = \{0, 1, 2\}$. $\{X_n : n = 0, 1, \ldots\}$ is a MC.

Transition prob. matrix:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{bmatrix} \end{array}$$

- P_{00} means start with 0 red & don't change color.
- P_{01} means start w/ 0 red & do change color

- P_{10} means start w/ 1 red, pick that red one (w/prob. 0.5) & change color (w/prob. 0.2) $= 0.5(0.2) = 0.1$

$$\bullet\ P_{11} = \underbrace{0.5}_{\text{pick red}}\underbrace{(0.8)}_{\text{don't change color}} \overset{\text{OR}}{+} \underbrace{0.5}_{\text{pick blue}}\underbrace{(0.8)}_{\text{don't change color}} = 0.8$$

$$\bullet\ P_{12} = \underbrace{(0.5)}_{\text{pick blue}}\underbrace{(0.2)}_{\text{change color}} = 0.1$$

$$P(5^{\text{th}} \text{ selection red} \mid \underbrace{X_0 = 2}_{\text{Initially both red}}) = \sum_{i=0}^{2} P(5^{\text{th}} \text{ sel. red} \mid X_4 = i)\underbrace{P(X_4 = i \mid X_0 = 2)}_{P_{2i}^{(4)}}$$

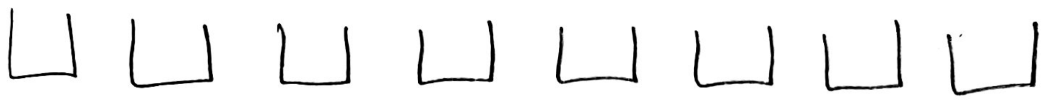
$$= \underbrace{0 \cdot P_{20}^{(4)}}_{i=0} + \underbrace{0.5 \cdot P_{21}^{(4)}}_{i=1} + \underbrace{1 \cdot P_{22}^{(4)}}_{i=2}$$

$$= 0 + 0.5\,(0.4352) + 1\,(0.4872)$$

$$= 0.7048$$

where

$$P^4 = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0.4872 & 0.4352 & 0.0776 \\ 1 & 0.2176 & 0.5648 & 0.2176 \\ 2 & 0.0776 & 0.4352 & 0.4872 \end{array}$$



Example 8: Eight urns

Suppose that balls are successively distributed among 8 urns, with each ball being equally likely to be put in any of these urns.

Question: What is the probability that there will be exactly 3 nonempty urns after 9 balls have been distributed?

Define (occupied)

X_n = number of nonempty urns after n balls have been distributed. all empty (only at time 0)

The desired probability is $P(X_9 = 3)$. $S = \{0, 1, 2, \ldots, 8\}$ all occupied w/ at least 1 ball

$\{X_n : n = 0, 1, \ldots\}$ is a MC w/ transition prob

$$P_{ii} = \frac{i}{8} = 1 - P_{i,i+1}$$

for $i = 0, 1, \ldots, 8$

e.g. $P_{11} = \frac{1}{8}$

$P_{12} = 1 - \frac{1}{8} = \frac{7}{8}$

Example 8: Eight urns (cont'd)

X_n starts with 0 occupied urns

Y_n starts with 1 occupied urn

$$\begin{aligned} P(X_9 = 3) &= P(X_9 = 3 | X_0 = 0) \\ &= P(X_9 = 3 | X_1 = 1, X_0 = 0) \\ &= P(X_9 = 3 | X_1 = 1) \\ &= P(Y_8 = 3 | Y_0 = 1) \quad \text{since } P_{01} = 1 \\ &= P_{13}^{(8)} \\ &= 0.00757 \end{aligned}$$

P =?

See next pg!

Example 8: Eight urns (cont'd)

can collapse these into a single state (since $P_{44} = 1$)

Another approach: States $4, 5, \ldots, 8$ cannot decrease to state 4.

redefine state 4: Whenever four or more of the urns are occupied, fix the state as 4.

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1/8 & 7/8 & 0 & 0 \\ 0 & 2/8 & 6/8 & 0 \\ 0 & 0 & 3/8 & 5/8 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\mathbf{P^4} = \begin{pmatrix} 0.0002 & 0.0256 & 0.2563 & 0.7178 \\ 0 & 0.0039 & 0.0952 & 0.9009 \\ 0 & 0 & 0.0198 & 0.9802 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cdot & \cdot & 0.2563 & \cdot \\ & & 0.0952 & \\ & & 0.0198 & \\ & & 0 & \end{pmatrix}$$

$\mathbf{P^8} = \mathbf{P^4 P^4}$

$P_{13}^{(8)} = 0.0002 \times 0.2563 + 0.0256 \times 0.0952 + 0.2563 \times 0.0198$
$+ 0.7178 \times 0 = 0.00756$

Example 9: Coin flips

Let N denote the number of flips until there is a run of **two** consecutive heads.

Question: Find $P(N \le 3)$.

Define $X_n = i$ consecutive heads at time n. , $S = \{0, 1, 2\}$

Intuitively, $P(N \le 3) = P(N = 2) + P(N = 3)$
$= HH\clubsuit + THH$
$= P_{01}P_{12}P_{22} + P_{00}P_{01}P_{12}$

This is the same as $P_{02}^{(3)}$. (All possibilities that $0 \to 2$ in 3 steps)

Thus, $X_3 = 2 \iff N \le 3$