§4.2 continued

n-step transition probability : $P_{ij}^{(n)} = P(X_{n+m} = j \mid X_m = i)$

Example 1 : Coin flips

Sequence of independent coin flips.
Let N denote the number of flips until there is a run of 2 consecutive heads : ... T H H

Q. Find $P(N \le 3)$.

Define $X_n = i$ consecutive heads at time n

$S = \{0, 1, 2\}$

Intuitively,

$$P(N \le 3) = P(N=2) \overset{\text{"OR"}}{+} P(N=3)$$

$$= HHE + THH$$

E means either H or T

$$= \underbrace{P_{01}P_{12}P_{22} + P_{00}P_{01}P_{12}}_{\text{This is equivalent to } P_{02}^{(3)}}$$

all possibilities of $0 \to 2$ transition in 3 steps.

In other words,

$X_3 = 2 \iff N \le 3$

Now, let N = # of flips until there is a run of $\underline{3}$ consecutive H's.

Q. Find $P(N \le 8)$.

Do we need to compute (No)

$$P(N \le 8) = P(N=3) + P(N=4) + \cdots + P(N=8) \quad ??$$

$N \le 8 \iff$ "There is a run of 3 $\overset{\text{consec.}}{\text{H's}}$ within the first 8 flips"

Define a MC on states $\{0, 1, 2, 3\}$

- For $i < 3$, state i means that we are currently in a run of i consecutive heads
- state $i = 3$ means that a run of 3 H's has already occurred

Then $X_n = i \iff i$ consecutive heads at time n

$$X_8 = 3 \iff N \le 8$$

The desired probability is $P_{03}^{(8)}$

Transition Prob. Matrix

(see Success Runs Example Lec 5)

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

assuming a fair coin: $p = q = \frac{1}{2}$

Thus, raising matrix $\mathbb{P}$ to the 8^{th} power, yields

$$\mathbb{P}^8 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 81/256 & 44/256 & 24/256 & 107/256 \\ 68/256 & 37/256 & 20/256 & 131/256 \\ 44/256 & 24/256 & 13/256 & 175/256 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \leftarrow P_{03}^{(8)} = 107/256 \approx 0.42$$

One more Question: $P(N=8)$?

$N=8 \iff$ "It takes 8 flips to obtain the FIRST run of 3 consec. H's"

Use $P(N=8) = P(N \leq 8) - P(N \leq 7)$

$$= P_{03}^{(8)} - P_{03}^{(7)}$$

OR

Note that $P(N=8) = \frac{1}{2} P_{02}^{(7)}$

If MC is in state 2 after 7 flips, then next flip is H with prob. $\frac{1}{2}$

Remark: Most probabilities we have computed thus far are conditional probs, such as $P_{ij}^{(n)}$.

To compute unconditional probabilities, need to know the dist'n of the initial state:

$$\alpha_i \equiv P(X_0 = i), \quad i \geq 0 \;\&\; \sum_{i=0}^{\infty} \alpha_i = 1$$

All unconditional probs. may be computed by conditioning on the initial state:

$$P(X_n=j) = \sum_{i=0}^{\infty} P(X_n=j \mid X_0=i)\,P(X_0=i) = \sum_{i=0}^{\infty} P_{ij}^{(n)} \alpha_i$$

§4.3 Classification of States

<u>def</u>: State j is <u>accessible</u> from state i if $\boxed{P_{ij}^{(n)} > 0}$ for some $n \geq 0$.

This means it is possible to go from i to j.

Denoted $i \to j$.

e.g. Simple random walk

$i \to j$ for any pair of states $i, j \in \mathbb{Z}$

RW reaches everywhere

<u>def</u>: Two states i & j <u>communicate</u> if $i \to j$ and $j \to i$.

Denoted by $i \leftrightarrow j$.

<u>Note</u>: Any state communicates with itself since, by def,

$$P_{ii}^{(0)} = P(X_0=i \mid X_0=i) = 1$$

The (equivalence) relation of communication satisfies the following:

(i) $i \leftrightarrow i \quad \forall\, i \in S$. (reflexive)

(ii) If $i \leftrightarrow j$, then $j \leftrightarrow i$ for $i,j \in S$. (symmetric)

(iii) If $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$ for $i,j,k \in S$. (transitive)

Proof :

(i) & (ii) are obvious from definition.

(iii) : Enough to show $i \to k$.

Since $i \leftrightarrow j$ and $j \leftrightarrow k$, $\exists$ integers n and m s.t.

$$P_{ij}^{(n)} > 0 \quad \text{and} \quad P_{jk}^{(m)} > 0 \text{ , respectively.}$$

By Chapman-Kolmogorov equations,

$$P_{ik}^{(n+m)} = \sum_{r \in S} P_{ir}^{(n)} P_{rk}^{(m)} \geq P_{ij}^{(n)} P_{jk}^{(m)} > 0 .$$

Thus, k is accessible from i ($i \to k$). Similarly, we can show $k \to i$. Hence i & k communicate.

def: If $i \leftrightarrow j$ (i & j communicate), then i & j are in the same class.

- The class of state i is $\{j : i \leftrightarrow j\}$.
- State space S is decomposed into 1 or more classes.
- All states in same class are accessible to each other.

def: If there is only 1 class, then all states communicate with each other & the Markov chain is said to be irreducible. *Important def!

* An irreducible MC can move freely from any state to any other state, in a sufficient # of steps, with positive probability.

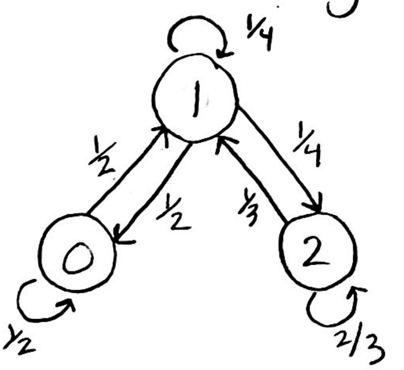
<u>def</u>: <u>Absorbing state</u> - once entered, there is no escape from an absorbing state.

$$P_{ii} = 1 \text{ and } P_{ij} = 0 \text{ for } i \neq j$$

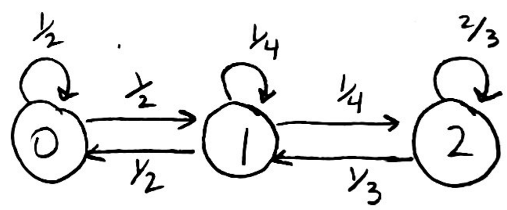
<u>Example 1</u>: MC defined on $S = \{0, 1, 2\}$ with transition prob. matrix

$$\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1/3 & 2/3 \end{bmatrix} \end{array}$$

Draw a transition diagram:



OR



Make sure the #'s on arrows leaving each state add to 1

same as rows summing to 1 in matrix P

Q. Is this MC irreducible?

Yes!

0 → 1 → 2 and

2 → 1 → 0

all states communicate!

1 class : $\{0, 1, 2\}$