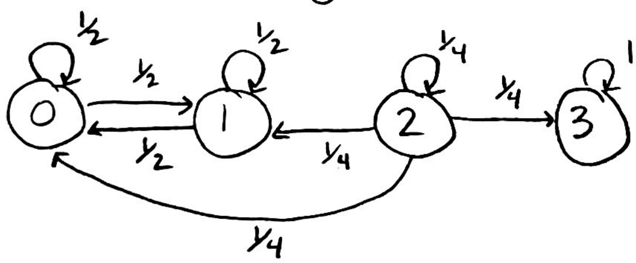
Example 2: MC on state space $S = \{0, 1, 2, 3\}$ with transition probability matrix

$$
\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{bmatrix}
1/2 & 1/2 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 \\
1/4 & 1/4 & 1/4 & 1/4 \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

absorbing state since $P_{33} = 1$

Draw a transition diagram



Q. Is this MC irreducible?

No! If a MC has an absorbing state, it is NOT irreducible (no other state is accessible from it).

Classes of states: $\{0, 1\}$, $\{2\}$, $\{3\}$

$0 \to 1$
$1 \to 0$
so $0 \leftrightarrow 1$

$2 \to 1$ but $1 \not\to 2$
$2 \to 0$ but $0 \not\to 2$
$2 \to 3$ but $3 \not\to 2$
so 2 is in a class by itself

absorbing states are always in a class of their own

§4.3 continued : Recurrence & Transience

<u>def</u>: Let f_{ii} be the probability that, starting in state i, the process will ever re-enter state i.

- If $f_{ii} = 1$, then state i is <u>recurrent</u>.
- If $f_{ii} < 1$, then state i is <u>transient</u>.

... Ross uses f_i

Suppose the process starts in a <u>recurrent state i</u>.

- With prob. 1, the process will eventually re-enter state i (since $f_{ii} = 1$).
- By def of MC, the process will be starting over again when it re-enters state i, and this will happen again and again ... in fact, <u>infinitely often</u>!

Suppose that state i is <u>transient</u>.

- Each time the process enters state i, there is a positive prob. ($1 - f_{ii}$, since $f_{ii} < 1$) that it will never enter state i again.
- So, starting in state i, the probability that the process will be in i for exactly n time periods is

$$f_{ii}^{\,n-1}(1 - f_{ii}) \quad \text{for } n \geq 1$$

$\Rightarrow$ # of time periods the process spends in transient state i follows a geometric dist'n w/mean $\frac{1}{1-f_{ii}} < \infty$

__Recall__: $X \sim \text{geometric}(p)$ has PMF $P(X=k) = (1-p)^{k-1} p$ for $k = 1, 2, \ldots$

$E[X] = \frac{1}{p}$

So in transient state i calculation earlier, $p = 1 - f_{ii}$ and hence the mean $= \frac{1}{1-f_{ii}}$ which is __finite__.

__Proposition__: State i is

- recurrent if and only if $\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$
- transient if and only if $\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$

SKIP in class

__Pf__: Let $I_n = \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{if } X_n \neq i \end{cases}$

$\sum_{n=1}^{\infty} I_n$ represents the # of periods that the process is in state i

$$E\left[\sum_{n=1}^{\infty} I_n \,\Big|\, X_0 = i\right] = \sum_{n=1}^{\infty} E[I_n \mid X_0 = i]$$

$$= \sum_{n=1}^{\infty} P(X_n = i \mid X_0 = i)$$

$$= \sum_{n=1}^{\infty} P_{ii}^{(n)}$$

- If i is recurrent, the process will re-enter state i again & again. Thus, $\sum_{n=1}^{\infty} I_n = \infty$ almost certainly.
- If i is transient, the expected value is $\frac{1}{1-f_{ii}} < \infty$.

* A transient state will only be visited a finite # of times
  ↳ hence the term "transient"

⇒ In a finite-state Markov chain, not all states can be transient.

Q. why? Suppose $S = \{0, 1, \dots, M\}$ and all states are transient.

- After a finite amount of time (say T_0), state 0 will never be visited.
- After a finite time T_1, state 1 will never be visited, and so on.

Thus, after a finite time ($\max\{T_0, T_1, \dots, T_M\}$), no states will be visited.

Contradiction!

$\boxed{\text{Thus, at least 1 of the states must be recurrent.}}$

Cor: If state i is recurrent and $i \leftrightarrow j$, then j is recurrent.

* Recurrence is a class property
* Transience is a class property
  ↳ If i is transient and $i \leftrightarrow j$, then j must also be transient. (If j were recurrent, then i would also be recurrent & hence could not be transient.)
  ^ by Cor above

$\boxed{\text{* All states of a finite irreducible MC are recurrent.}}$

* If i is recurrent and j is transient, then $i \nrightarrow j$
(j is NOT accessible from i)

↳ Suppose $i \to j$. If $j \to i$, then $i \leftrightarrow j$ and i & j are in same class. By class property, j should be recurrent. Contradiction! Thus, $j \nrightarrow i$.
So, $i \to j$ but $j \nrightarrow i$. This means that $\exists$ positive probability of leaving i (and going to j) and never coming back to i. Contradiction, since i is recurrent. Hence, $i \nrightarrow j$.

* Starting from a recurrent state, MC cannot enter a transient state. The converse IS Possible!

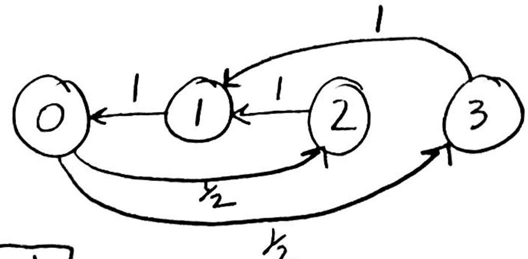
$$\text{transient state} \to \text{recurrent state}$$

<u>Remark</u>: When a MC is irreducible, we talk about the recurrence or transience of the chain (as opposed to subsets of states)

<u>Example 1</u>: Consider a Markov chain on states $S = \{0, 1, 2, 3\}$ and

$$\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Determine which states are transient & which are recurrent.

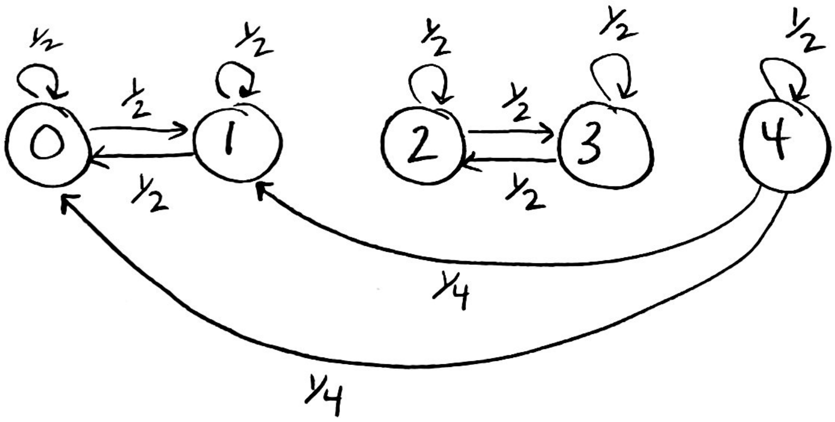


All states communicate
⇒ irreducible MC
⇒ $\boxed{\text{all states recurrent}}$ since state space is finite

Example 2: Consider a Markov chain on $S = \{0,1,2,3,4\}$ with transition prob. matrix

$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 1 & 1/2 & 1/2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1/2 & 1/2 & 0 \\ 3 & 0 & 0 & 1/2 & 1/2 & 0 \\ 4 & 1/4 & 1/4 & 0 & 0 & 1/2 \end{array}$$

Determine which states are transient & which are recurrent.



Recurrent classes: $\{0,1\}$ and $\{2,3\}$

$0 \to 1$ $\quad P_{01} > 0$ $\qquad$ $2 \to 3$ $\quad P_{23} > 0$

$1 \to 0$ $\quad P_{10} > 0$ $\qquad$ $3 \to 2$ $\quad P_{32} > 0$

so, 0 & 1 communicate $\qquad$ so, 2 & 3 communicate

Transient class: $\{4\}$

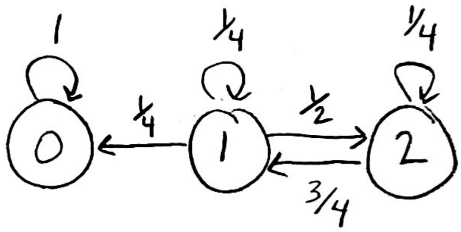
$4 \to 1$ $\quad P_{41} > 0$ but once enter state 1, cannot leave class $\{0,1\}$ - recurrent

Likewise, $4 \to 0$ $P_{40} > 0$ but once enter state 0, cannot leave class $\{0,1\}$ - recurrent

Even though $4 \to 4$ $(P_{44} > 0)$, the MC will only stay in state 4 for a finite # of time steps & then will leave (never to return) $\Rightarrow$ $\{4\}$ is transient class

<u>Example 3</u>: MC on states $S = \{0,1,2\}$ with

$$\mathbb{P} = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1/4 & 1/2 \\ 0 & 3/4 & 1/4 \end{bmatrix} \end{array}$$



Determine which states are recurrent & which are transient.

<u>Note</u>: State 0 is an absorbing state $\Rightarrow$ recurrent

Recurrent class: $\{0\}$

Transient class: $\{1,2\}$

since $1 \to 0$ but $0 \not\to 1$

$1 \to 2$ and $2 \to 1$ so if one is transient, then they both are (class property)

$2 \to 1 \to 0$ but $0 \not\to 2$