[for proof of SLLN version 1, see Rosenthal]

## Eliminating the moment conditions

def: A collection of random variables $\{X_\alpha\}_{\alpha \in I}$ are identically distributed if for any Borel measurable function $f : \mathbb{R} \to \mathbb{R}$, the expected value $E[f(X_\alpha)]$ does not depend on $\alpha$, i.e. is the same for all $\alpha \in I$.

Remark: It follows that $\{X_\alpha\}_{\alpha \in I}$ are identically distributed iff $\forall x \in \mathbb{R}$, $P(X_\alpha \leq x)$ does not depend on $\alpha$.

def: A collection of RVs $\{X_\alpha\}_{\alpha \in I}$ are i.i.d. if they are independent and identically distributed.

Thm [Strong LLNs Version 2]: Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables, each having finite mean $\mu$. Then

$$P\left(\lim_{n \to \infty} \frac{1}{n}(X_1 + \cdots + X_n) = \mu\right) = 1.$$

i.e. $\frac{S_n}{n} \xrightarrow{a.s.} \mu$ where $S_n = X_1 + \cdots + X_n$

Pf: To begin, we assume that $X_i \geq 0$. If not, we can consider separately $X^+$ and $X^-$. We set

$$Y_i = X_i \mathbb{1}_{\{X_i \leq i\}}, \quad \text{i.e. } Y_i = X_i \text{ unless } X_i \text{ exceeds } i$$

(in which case $Y_i = 0$).

"truncation argument"

Then since $0 \leq Y_i \leq i$, we have that

$$E[Y_i^k] \leq i^k < \infty.$$

Also, note that the $\{Y_i\}$ are independent (function of $X_i$'s & $X_i$'s indep).

Furthermore, since the $X_i$'s are i.i.d.

$$E[Y_i] = E[X_i \mathbb{1}_{\{X_i \leq i\}}] = E[X_1 \mathbb{1}_{\{X_1 \leq i\}}] \nearrow E[X_1] = \mu$$

as $i \to \infty$, by the MCT.

Now set $S_n = \sum_{i=1}^n X_i$ and $S_n^* = \sum_{i=1}^n Y_i$. We compute

$$\begin{aligned}
Var(S_n^*) &= Var(Y_1) + \cdots + Var(Y_n) && \text{since } \{Y_i\} \text{ indep} \\
&\leq E[Y_1^2] + \cdots + E[Y_n^2] && \text{(always true)} \\
&= E[X_1^2 \mathbb{1}_{\{X_1 \leq 1\}}] + \cdots + E[X_n^2 \mathbb{1}_{\{X_n \leq n\}}] && \text{by def} \\
(5.4.5) \longrightarrow &\leq n\, E[X_1^2 \mathbb{1}_{\{X_1 \leq n\}}] && \text{why?} \\
&\leq n \cdot n^2 = n^3 < \infty
\end{aligned}$$

$E[Y_i^k] \leq i^k < \infty$ from above & $X_i$'s are i.i.d so $P(X_i \leq n)$ doesn't depend on $i$

We now choose $\alpha > 1$ (later we will let $\alpha \searrow 1$) and set $u_n = \lfloor \alpha^n \rfloor$. Then $u_n \le \alpha^n$. Furthermore, since $\alpha^n > 1$, it follows that $\underbrace{u_n \geq \frac{\alpha^n}{2}}_{\text{clarify!}} \Rightarrow \frac{1}{u_n} \le \frac{2}{\alpha^n}$.

Hence, for any $x > 0$,

$$\sum_{\substack{n \\ u_n \geq x}} \frac{1}{u_n} \le \sum_{\substack{n \\ \alpha^n \geq x}} \frac{1}{u_n} \le \sum_{\substack{n \\ \alpha^n \geq \alpha}} \frac{2}{\alpha^n} \le \sum_{k = \log_\alpha x}^{\infty} \frac{2}{\alpha^k} = \frac{2/x}{1 - \frac{1}{\alpha}} \qquad (5.4.6)$$

since $u_n \le \alpha^n$ — since $\frac{1}{u_n} \le \frac{2}{\alpha^n}$ — why? $\alpha^{\log_\alpha x} = x$ — geometric sum

Lec 12 (1a)

Now (heart of the proof), for any $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P\left( \left| \frac{S^*_{u_n} - E[S^*_{u_n}]}{u_n} \right| \geq \varepsilon \right)$$

$$= \sum_{n=1}^{\infty} P\left( \left| \frac{S^*_{u_n}}{u_n} - E\left[ \frac{S^*_{u_n}}{u_n} \right] \right| \geq \varepsilon \right)$$

$$\le \sum_{n=1}^{\infty} \frac{Var(S^*_{u_n}/u_n)}{\varepsilon^2} \quad \text{by Chebyshev's Ineq.}$$

$$= \sum_{n=1}^{\infty} \frac{Var(S^*_{u_n})}{u_n^2 \varepsilon^2} \quad \text{by property of variance}$$

$$\left[ Var(aX) = a^2 Var(X) \right]$$

$$\le \sum_{n=1}^{\infty} \frac{u_n \, E[X_1^2 \, \mathbb{1}_{\{X_1 \le u_n\}}]}{u_n^2 \, \varepsilon^2} \quad \text{by } (5.4.5)$$

$$= \frac{1}{\varepsilon^2} E\left[X_1^2 \sum_{n=1}^{\infty} \frac{1}{u_n} \mathbb{1}_{\{u_n \ge X_1\}}\right] \quad \text{by countable linearity of } E(\cdot)$$

$$\le \frac{1}{\varepsilon^2} E\left[X_1^2 \left(\frac{2/X_1}{1-\frac{1}{\alpha}}\right)\right] \quad \text{by } (5.4.6)$$

$$= \frac{2}{\varepsilon^2(1-\frac{1}{\alpha})} E\left[X_1^2 \cdot \frac{1}{X_1}\right] = \frac{2}{\varepsilon^2(1-\frac{1}{\alpha})} E[X_1]$$

$$= \frac{2\mu}{\varepsilon^2(1-\frac{1}{\alpha})} < \infty.$$

(since $E[X_1]$ is assumed to be finite)

This finiteness is key!

$\forall \varepsilon > 0$, if $\sum_n P(|Z_n - Z| \ge \varepsilon) < \infty$, then $Z_n \xrightarrow{a.s.} Z$

It now follows from Cor 5.2.2.

that
$$\left\{ \frac{S_{u_n}^* - E[S_{u_n}^*]}{u_n} \right\} \xrightarrow{a.s.} 0 \quad \text{as } n \to \infty. \qquad (5.4.7)$$

To complete the proof, need to (1) replace $\frac{E[S_{u_n}^*]}{u_n}$ by $\mu$, (2) replace $S_{u_n}^*$ by $S_{u_n}$, and finally (3) replace the index $u_n$ by the general index $k$.

① First, since $E[Y_i] \to \mu$ as $i \to \infty$ and since $u_n \to \infty$ as $n \to \infty$, it follows immediately that

$$\frac{E[S^*_{u_n}]}{u_n} \to \mu \quad \text{as } n \to \infty.$$

Then by (5.4.7), it follows that $\left\{ \frac{S^*_{u_n}}{u_n} \right\} \xrightarrow{a.s.} \mu$ as $n \to \infty$. (5.4.8)

② Note that

$$\sum_{k=1}^{\infty} P(X_k \neq Y_k) = \sum_{k=1}^{\infty} P(X_k > k) \leq \sum_{k=1}^{\infty} P(X_k \geq k)$$

$$= \underbrace{\sum_{k=1}^{\infty} P(X_1 \geq k)}_{\text{since } X_i\text{'s are iid}} = E[\lfloor X_1 \rfloor] \leq E[X_1] = \mu < \infty.$$

by the prop. from Lec 9 (4.2.9 in R)

By Borel-Cantelli,

$P(X_k \neq Y_k \text{ i.o.}) = 0$ so that $P(X_k = Y_k \text{ a.a.}) = 1$.

almost always

on all but a finite set

It follows that, w.p. 1 (i.e. a.s.), as $n \to \infty$ the limit of $\frac{S^*_{u_n}}{u_n}$ coincides with limit of $\frac{S_{u_n}}{u_n}$.

Hence, (5.4.8) $\Rightarrow$ $\left\{ \frac{S_{u_n}}{u_n} \right\} \xrightarrow{a.s.} \mu$

(3) Finally, for an arbitrary index $k$, we can find $n = n_k$ s.t. $u_n \le k < u_{n+1}$. But then

$$\frac{u_n}{u_{n+1}} \cdot \frac{S_{u_n}}{u_n} = \frac{S_{u_n}}{u_{n+1}} \le \frac{S_k}{k} \le \frac{S_{u_{n+1}}}{u_n} = \frac{S_{u_{n+1}}}{u_{n+1}} \cdot \frac{u_{n+1}}{u_n}.$$

Now as $k \to \infty$, we have $n = n_k \to \infty$ so that

$$\frac{u_n}{u_{n+1}} \to \frac{1}{\alpha} \quad \text{and} \quad \frac{u_{n+1}}{u_n} \to \alpha.$$

Hence, for any $\alpha > 1$ and $\delta > 0$, with prob. 1 we have $\frac{\mu}{(1+\delta)\alpha} \le \frac{S_k}{k} \le (1+\delta)\alpha\mu$ for all sufficiently large $k$. For any $\varepsilon > 0$, choosing $\alpha > 1$ and $\delta > 0$ s.t. $\frac{\mu}{(1+\delta)\alpha} > \mu - \varepsilon$ and $(1+\delta)\alpha\mu < \mu + \varepsilon$

$$\Rightarrow P\left(\left|\frac{S_k}{k} - \mu\right| \ge \varepsilon \text{ i.o}\right) = 0.$$

Hence, by Lemma (5.2.1) [from Lec 10], we have that as $k \to \infty$,

$$\frac{S_k}{k} \xrightarrow{a.s.} \mu$$ as required. ⊠

Now by Prop 5.2.3 (Rosenthal) – $Z_n \xrightarrow{a.s.} Z \Rightarrow Z_n \xrightarrow{P} Z$

we immediately get a weaker version of LLNs:

* Cor [Weak LLNs version 2]:

Let $X_1, X_2, \dots$ be a sequence of i.i.d. random variables, each having finite mean $\mu$. Then $\forall \varepsilon > 0$,

$$\lim_{n\to\infty} P\left(\left|\frac{1}{n}\underbrace{(X_1 + \cdots + X_n)}_{S_n} - \mu\right| \geq \varepsilon\right) = 0$$

i.e. $\frac{S_n}{n} \xrightarrow{P} \mu$ (convergence in probability).

Pf: SLLNs version 2 combined with Prop 5.2.3.