Given any Borel-measurable function f (aka <u>density fct</u>) s.t. $f \geq 0$ and $\int_{-\infty}^{\infty} f(t)\, d\lambda(t) = 1$, we can define

a distribution function (law) μ by

$$\boxed{\mu(B) = \int_{-\infty}^{\infty} f(t)\, \mathbb{1}_B(t)\, d\lambda(t)}, \quad B \in \mathcal{B} \quad \text{(Borel set)}$$

[Sometimes write this as $\mu(B) = \int_B f\, d\lambda$ or $\int_B f(t)\, d\lambda(t)$.]

[$\int_B d\mu(t) = \int_B f(t)\, d\lambda(t) \quad \forall\, B \in \mathcal{B} \quad \Leftrightarrow \quad d\mu(t) = f(t)\, d\lambda(t)$

$\frac{d\mu}{d\lambda} = f$

"with respect to"

μ is absolutely continuous w.r.t. λ

f is the density for μ w.r.t. λ]

<u>Prop 6.2.3</u>: Suppose μ has density f w.r.t. λ. Then for any Borel-measurable function $g: \mathbb{R} \to \mathbb{R}$,

$$E_\mu[g] \equiv \int_{-\infty}^{\infty} g(t)\, d\mu(t) = \int_{-\infty}^{\infty} g(t) f(t)\, d\lambda(t)$$

provided either side is well-defined.

Now it is possible to do explicit computations with absolutely continuous RVs :

e.g. $X \sim N(0,1)$

$$E[X] = \int t \, d\mu_N(t) = \int t \, \phi(t) \, d\lambda(t)$$

$$= \int_{-\infty}^{\infty} t \, \phi(t) \, dt$$

$$= \int_{-\infty}^{\infty} t \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$$

More generally,

$$E[g(X)] = \int g(t) \, d\mu_N(t) = \int g(t) \, \phi(t) \, d\lambda(t)$$

$$= \int_{-\infty}^{\infty} g(t) \, \phi(t) \, dt$$

for any Riemann-integrable function g

Stochastic Processes

(Ref: §7 Rosenthal, §7 Billingsley)

def: A discrete time stochastic process is a sequence of random variables $X_0, X_1, X_2, \dots$ defined on some probability space $(\Omega, \mathcal{F}, P)$.

Typically, the X_n's are not independent.
Think of the index "n" as representing time.

$\Rightarrow$ X_n is the value of a random quantity at time n.

Example: Infinite fair coin tossing $(r_1, r_2, \dots)$ where $r_i = 0$ or 1 with prob. $\frac{1}{2}$. Say 0 = Tails, 1 = Heads.

Set $X_0 = 0$; $X_n = r_1 + \cdots + r_n$, $n \geq 1$

↗ # of heads obtained by time n

SKIP ?

Thm [Existence]: Let $\mu_1, \mu_2, \dots$ be any sequence of Borel probability measures on $\mathbb{R}$. Then there exists a prob. space $(\Omega, \mathcal{F}, P)$ and RVs $X_1, X_2, \dots$ defined on $(\Omega, \mathcal{F}, P)$ s.t. $\{X_n\}$ are independent and $\mathcal{L}(X_n) = \mu_n$.

Gambling & Gambler's Ruin

Let $Z_1, Z_2, \ldots$ be i.i.d. RVs s.t.

$$P(Z_i = 1) = p \quad \text{and} \quad P(Z_i = -1) = 1-p \quad \text{for some fixed } p,\ 0<p<1.$$

Let $X_n = a + Z_1 + \cdots + Z_n$ with $X_0 \equiv a$ for some $a \in \mathbb{N}$.

Interpret X_n as gambler's fortune (in \$) at time n when repeatedly making \$1 bets.

Then $\{X_n\}$ is a stochastic process known as a <u>simple random walk</u>.

- Player begins with \$a
- At each time step, either wins \$1 w/prob p or loses \$1 with prob. $1-p$.

Q. What is the distribution of X_n ?

$$P(X_n = a+k) = 0 \quad \text{unless} \quad -n \le k \le n \text{ with } n+k \text{ even}$$

Say $a = 2$.

$$X_1 = 2 + \underbrace{Z_1}_{k} \begin{cases} 2+1 & \text{if } Z_1 = +1 \\ 2-1 & \text{if } Z_1 = -1 \end{cases} \quad \begin{matrix} n=1 \\ n+k = 1+1 \text{ (even)} \\ n+k = 1-1 \text{ (even)} \end{matrix}$$

$$X_2 = 2 + \underbrace{Z_1 + Z_2}_{k} \qquad k = -2, 0, 2 \qquad n=2 \qquad \text{so } n+k \text{ is even}$$

For such k, there are $\binom{n}{\frac{n+k}{2}}$ different possible sequences $Z_1, Z_2, \ldots$ s.t. $X_n = a+k$,

i.e. all seqs consisting of $\frac{n+k}{2}$ $+1$'s and $\frac{n-k}{2}$ -1's.

e.g. $X_2 = 2 + Z_1 + Z_2$

Z_1	Z_2
-1	-1
-1	$+1$
$+1$	-1
$+1$	$+1$

say $k=2$

$n+k = 2+2 = 4$

$\frac{n+k}{2} = \frac{4}{2} = 2$ $+1$'s

$\frac{n-k}{2} = \frac{0}{2} = 0$ -1's

Each such sequence has probability $p^{\frac{n+k}{2}}(1-p)^{\frac{n-k}{2}}$

Thus,

$$\boxed{P(X_n = a+k) = \binom{n}{\frac{n+k}{2}} p^{\frac{n+k}{2}} (1-p)^{\frac{n-k}{2}}, \quad -n \le k \le n \ \& \ n+k \text{ even}}$$

& 0 otherwise.

Gambler's Ruin Problem:

Suppose that $0 < \underset{\substack{\uparrow \\ \text{initial capital}}}{a} < \underset{\substack{\uparrow \\ \text{goal}}}{c}$, and let

$\tau_0 = \inf\{n \ge 0 : X_n = 0\}$

$\tau_c = \inf\{n \ge 0 : X_n = c\}$ be the first hitting time of 0 and c, respectively.

Gambler's ruin question is : what is $P(\tau_c < \tau_0)$?

In words, what is the probability the gambler will get rich (reach \$c) before going broke (reach \$0)?

(Note: $\{\tau_c < \tau_0\}$ includes the case $\tau_0 = \infty$ while $\tau_c < \infty$ but not the case $\tau_c = \tau_0 = \infty$.)

Solving this is not straightforward.

There's a nice trick!

Set $s(a) = P(\tau_c < \tau_0)$.

write dependence on a explicitly allows us to vary a & relate s(a) to s(a-1) & s(a+1)

For $1 \leq a \leq c-1$, we have

$$
\begin{aligned}
s(a) &= P(\tau_c < \tau_0) \\
&= P(Z_1 = -1, \tau_c < \tau_0) + P(Z_1 = 1, \tau_c < \tau_0) \\
&= (1-p)\, s(a-1) + p\, s(a+1)
\end{aligned}
$$

since Z_i's indep. & $Z_2, Z_3, \ldots$ is a probabilistic replica of $Z_1, Z_2, \ldots$

Further, $s(0) = 0$ & $s(c) = 1$ by definition

If a=0, can never play

If a=c, clearly you reach \$c before \$0.