Solve system of c-1 equations for the c-1 unknowns:

$s(1), s(2), \ldots, s(c-1)$ ← HW problem

To obtain

Note: $(q = 1-p)$

$$\begin{cases} s(a) = P(\tau_c < \tau_0) = \dfrac{1-\left(\frac{q}{p}\right)^a}{1-\left(\frac{q}{p}\right)^c} & \text{for } p \neq \frac{1}{2} \\[2ex] s(a) = \dfrac{a}{c} & \text{for } p = \frac{1}{2} \end{cases}$$

Specific Example:

Suppose you start with \$9700 $(a = 9700)$ and your goal is to win \$10,000 before going broke $(c = 10000)$.

If $p = \frac{1}{2}$, then prob. of success $= \frac{a}{c} = 0.97$
$P(\tau_c < \tau_0)$

very high!

If $p = 0.49 < \frac{1}{2}$, then prob. of success is

$$\frac{1-\left(\frac{0.51}{0.49}\right)^{9700}}{1-\left(\frac{0.51}{0.49}\right)^{10000}} \approx 0.0000061$$

OR

1 chance in 163,000.

dramatic change!
only small disadv. on each bet...

Q. What is the probability the gambler will go broke if they never stop gambling? i.e. $P(\tau_0 < \infty)$

Let $H_c = \{\tau_0 < \tau_c\}$. Then $\{H_c\}$ is increasing up to $\{\tau_0 < \infty\}$ as $c \to \infty$. So by continuity of prob.,

$$P(\tau_0 < \infty) = \lim_{c \to \infty} P(H_c)$$

$$= \lim_{c \to \infty} r(a) \qquad \text{where } r(a) = P(\tau_0 < \tau_c)$$

($\&$ note that $r(a) = s(c-a)$)

$$= \begin{cases} 1 & \text{if } p \leq \frac{1}{2} \\ \left(\frac{q}{p}\right)^a & \text{if } p > \frac{1}{2} \end{cases}$$

see R for details

Thus, if $p \leq \frac{1}{2}$, the gambler is certain to go broke!

Hence, "Gambler's ruin".

Only chance for success occurs if $p > \frac{1}{2}$.

Suppose now that the gambler is allowed to choose how much to bet each time. That is,

Choose RVs W_n s.t. fortune at time n is

$$X_n = a + W_1 Z_1 + \cdots + W_n Z_n$$

with $\{Z_n\}$ as before, $\&$ $W_n \geq 0$

Note that $X_n = X_{n+1} + W_n Z_n$, W_n & Z_n are indep.

and $E[Z_n] = p-q$.

$$\Rightarrow E[X_n] = E[X_{n-1}] + (p-q)E[W_n]$$

Then

(i) If $p=\frac{1}{2}$, $E[X_n] = E[X_{n-1}] = \cdots = E[X_0] = a$

and $\lim_{n\to\infty} E[X_n] = a$.

(ii) If $p \le \frac{1}{2}$, $E[X_n] \le E[X_{n-1}] \le \cdots \le E[X_0] = a$

(since $W_n \ge 0 \Rightarrow E[W_n] \ge 0$)

and $\lim_{n\to\infty} E[X_n] \le a$.

(iii) If $p \ge \frac{1}{2}$, $E[X_n] \ge E[X_{n-1}] \ge \cdots \ge E[X_0] = a$

and $\lim_{n\to\infty} E[X_n] \ge a$.

With "clever betting" (e.g. double until you win), a gambler can cheat fate. However, this requires infinite capital.
(always win)

If capital is finite (as in reality) and $p \le \frac{1}{2}$, then on average they will always lose \$ by the

<u>Bounded Convergence Thm</u>.

Thm [Bounded Convergence]: Let $\{X_n\}$ be a sequence of RVs with $\lim_{n\to\infty} X_n = X$. Suppose $\exists\ K \in \mathbb{R}$ s.t. $|X_n| \le K \ \forall\, n \in \mathbb{N}$ (i.e. X_n are uniformly bounded).

Then $E[X] = \lim_{n\to\infty} E[X_n]$.

Discrete Markov Chains

(Ref: R/B § 8)

MCs illustrate a clear connection between probability & measure. We will develop their basic properties in a measure-theoretic setting.

First, we consider the general notion of a discrete-time, discrete-space, time-homogeneous Markov chain.

A <u>Markov chain</u> is characterized by 3 parts:

- state space S (finite or countable set)
- initial distribution $\{\alpha_i\}_{i \in S}$, $\alpha_i \geq 0 \ \forall i$ (non-neg.) and α_i's sum to 1
- transition probabilities $\{p_{ij}\}_{i,j \in S}$, $p_{ij} \geq 0 \ \forall i,j$ and $\sum_{j \in S} p_{ij} = 1$ for each $i \in S$

OR in matrix form

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & & & \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

(then the <u>rows</u> sum to 1)

"Transition Probability Matrix"

Intuitively, a Markov chain (MC) represents the random motion of some particle moving around the space S.

α_i = probability that the particle starts at point i

P_{ij} = prob. that the particle moves from state i to state j in the next step

Formal Def:

A Markov chain is a sequence of random variables $X_0, X_1, X_2, \ldots$ taking values in S (sample space) s.t.

$$P(X_{n+1} = j \mid X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n (= i)) \qquad (*)$$
$$= P(X_{n+1} = j \mid X_n = i) = P_{ij}$$

for every n and every sequence $i_0, i_1, \ldots, i_n$ in S for which $P(X_0 = i_0, \ldots, X_n = i_n) > 0$.

Eqn $(*)$ above shows that $\{X_n : n \geq 0\}$ satisfies the Markov Property: the conditional distribution of the next state X_{n+1} given the present state X_n doesn't depend on the past $X_0, X_1, \ldots, X_{n-1}$.