Show (iv) $\Rightarrow$ (ii) :

Suppose to the contrary of (ii) that $f_{ij} < 1$ for some $i, j \in S$, i.e. ~~some state is not recurrent~~.

Then $1 - f_{jj} \geq \underbrace{P_j(\tau_i < \tau_j)}_{\substack{>0 \\ \text{by irreducibility}}} \underbrace{(1 - f_{ij})}_{>0 \text{ by assumption}} > 0.$

$\Rightarrow f_{jj} < 1$

$\Rightarrow P_j(X_n = j \text{ i.o.}) = 0$ and by Thm 8.2.1

we have $\sum_n p_{jj}^{(n)} < \infty$ which contradicts (iv).

<u>Some Details</u>

$$P_{ij}^{(m+n+r)} = \sum_{k,l} P_{ik}^{(m)} P_{kl}^{(n)} P_{lj}^{(r)} \geq P_{ik}^{(m)} P_{kl}^{(n)} P_{lj}^{(r)}$$

Q. $\underline{\sum_n P_{ij}^{(n)}} \geq P_{ik}^{(m)} P_{lj}^{(r)} \underbrace{\sum_n P_{kl}^{(n)}}_{\text{given in (iii)}} = \infty$

Why ?

at some point this goes beyond $m+n+r$. So this sum $\sum_n P_{ij}^{(n)} \geq P_{ij}^{(m+n+r)}$

Other Details

Recall that i.o. means "infinitely often"

$$P_i(X_n = j \text{ i.o.}) = \begin{cases} 0 & \text{if } f_{ij} < 1 \\ 1 & \text{if } f_{ij} = 1 \end{cases}$$

Setting $i=j$ gives

$$P_i(X_n = i \text{ i.o.}) = \begin{cases} 0 & \text{if } f_{ii} < 1 \\ 1 & \text{if } f_{ii} = 1 \end{cases}$$

---

n^{th} step transition probability:

$$p_{ij}^{(n)} = P(X_{m+n} = j \mid X_m = i)$$

$$= \sum_{k_1, \dots, k_{n-1}} p_{ik_1} p_{k_1 k_2} \cdots p_{k_{n-1} j}$$

Also, $p_{ij}^{(n)}$ is the $(i,j)^{th}$ entry in matrix P^n

$$p_{ij}^{(m+n)} = \sum_k p_{ik}^{(m)} p_{kj}^{(n)}, \quad \sum_j p_{ij}^{(n)} = 1$$

(still has row sums $=1$)

Apply Thm 8.2.3 :

For example, symmetric RW – since $\sum_n p_{ii}^{(n)} = \infty$, this Thm says that from any state i, the walk will eventually reach any other state j (w/prob. 1).

* All states recurrent

Also, with prob. 1,

$$\limsup_n X_n = \infty \quad \text{and} \quad \liminf_n X_n = -\infty.$$

Stationary Distributions & Convergence (R §8.3)

def: Given a Markov chain on a state space S with transition probability matrix $P = \{p_{ij}\}_{i,j \in S}$, let $\pi = \{\pi_i\}_{i \in S}$ be a distribution on S (i.e. $\pi_i \geq 0 \ \forall i \in S$ and $\sum_{i \in S} \pi_i = 1$). Then π is a stationary distribution if

$$\boxed{\sum_{i \in S} \pi_i p_{ij} = \pi_j} \quad \text{for } j \in S.$$

In matrix form: $\pi P = \pi$

[Equivalently, $\pi P^n = \pi$ for any $n \in \mathbb{N}$]

Some intuition:

- Suppose we start the Markov chain in the distribution $\{\pi_i\}_{i\in S}$, i.e. $P(X_0=i)=\pi_i \ \forall i\in S$.

- Then 1 time step later the distribution is still $\{\pi_i\}_{i\in S}$, i.e. $P(X_1=i)=\pi_i \ \forall i\in S$.

Hence, the term "stationary" distribution.

- By induction, it follows that $P(X_n=i)=\pi_i \ \forall i\in S$

  for any # of steps $n\in\mathbb{N}$

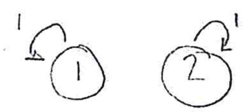
Q. Is it true that $\lim_{n\to\infty} P_i(X_n = j) = \pi_j$ for any $i, j \in S$?

A. Yes for some MCs, but not true in general.

Conditions for this to be true:

- MC needs to be <u>irreducible</u> & <u>aperiodic</u> &
- MC needs to have a stationary dist'n → we'll discuss existence in next subsection

<u>Example 1</u>:



$S = \{1, 2\}$ $\quad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

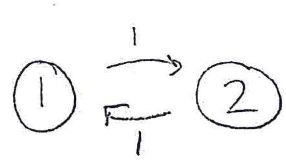
This MC never moves!
⇒ Any distribution is stationary! $\pi P = \pi$
e.g. $\pi_1 = \pi_2 = \frac{1}{2}$

But $P_1(X_n = 1) = 1 \ \forall n \in \mathbb{N}$
which does not approach $\frac{1}{2}$.

Thus, this MC is reducible (i.e. NOT irreducible) ← more than 1 class of states
which is what doesn't allow the MC to converge to a stationary dist'n.

<u>Example 2</u>:

$S = \{1, 2\}$ $\quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



$\left(\begin{aligned} &\pi P = \pi \\ &\Rightarrow \pi_1 = \pi_2 = \tfrac{1}{2} \end{aligned} \right)$

stationary distribution: $\pi_1 = \pi_2 = \frac{1}{2}$, unique

This MC is irreducible (only 1 class of states, all communicate w/each other)

But $P_1(X_n = 1) = \begin{cases} 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$

Again $P_1(X_n = 1) \not\to \frac{1}{2}$ as $n \to \infty$.

Why? Because the MC is periodic (period $= 2$).

<u>def</u>: Given a MC defined on S with transition prob. matrix P, and a state $i \in S$, the <u>period</u> of i is the greatest common divisor of the set $\{n \geq 1 : P_{ii}^{(n)} > 0\}$.

i.e. GCD of the times at which it is possible to travel from i to i.

e.g. period of i $= 2$ means only possible to return to i in an even # of steps, as in Ex 2.

<u>def</u>: A MC is <u>aperiodic</u> if the period of each state is 1.

<u>Lemma 8.3.5</u>: Let $i, j \in S$ for a MC and suppose that $f_{ij} > 0$ and $f_{ji} > 0$ (i.e. i & j <u>communicate</u>). Then the periods of i & j are equal.

Cor 8.3.7 : If a MC is irreducible, then all of its states have the same period.

→ For an irreducible MC, it suffices to check aperiodicity at a single state.

Lemma 8.3.8 : If a MC is irreducible, and has a stationary distribution π, then it is recurrent.

NOTE : Converse of this lemma is NOT true!

e.g. simple symmetric RW (1-dim) – irred., recurrent MC but doesn't have a stationary distn.

Now for main MC convergence theorem :

* Thm 8.3.10 : If a MC is irreducible & aperiodic & has a stationary distribution π, then $\forall$ states i & j we have $\lim_{n\to\infty} P_{\underline{i}}(X_n = j) = \pi_j$.

Cor 8.3.11 : (Same set up as Thm 8.3.10) – regardless of the initial distribution, for all $j \in S$, $\lim_{n\to\infty} P(X_n = j) = \pi_j$.

no conditioning on $X_0 = i$

Pf of Cor 8.3.11 : Let $\{X_n : n \geq 0\}$ be an irreducible & aperiodic MC that has stationary distribution $\pi = \{\pi_i\}_{i \in S}$. Then

$$\lim_{n\to\infty} P(X_n = j) = \lim_{n\to\infty} \sum_{i \in S} P(X_0 = i, X_n = j)$$

$$= \lim_{n\to\infty} \sum_{i\in S} P(X_0 = i)\, P(X_n = j \mid X_0 = i)$$

$$= \lim_{n\to\infty} \sum_{i\in S} \alpha_i P_i(X_n = j)$$

$$= \sum_{i \in S} \alpha_i \lim_{n\to\infty} P_i(X_n = j)$$

$$= \underbrace{\sum_{i\in S} \alpha_i}_{1} \pi_j \quad \text{by Thm 8.3.10 since } \pi \text{ is a stationary distribution}$$

$$= \pi_j$$

which gives the desired result.