<u>Pf of Cor 8.3.11</u>: Let $\{X_n : n \geq 0\}$ be an irreducible & aperiodic MC that has stationary distribution $\pi = \{\pi_i\}_{i \in S}$. Then

$$\lim_{n\to\infty} P(X_n = j) = \lim_{n\to\infty} \sum_{i \in S} P(X_0 = i, X_n = j)$$

$$= \lim_{n\to\infty} \sum_{i\in S} P(X_0 = i)\, P(X_n = j \mid X_0 = i)$$

$$= \lim_{n\to\infty} \sum_{i\in S} \alpha_i\, P_i(X_n = j)$$

$$= \sum_{i\in S} \alpha_i \lim_{n\to\infty} P_i(X_n = j) \quad \text{by M-test (see appendix in R)}$$

$$= \underbrace{\sum_{i\in S} \alpha_i}_{1}\, \pi_j \quad \text{by Thm 8.3.10 since } \pi \text{ is a stationary distribution}$$

$$= \pi_j$$

which gives the desired result.

Existence of Stationary Distributions (R §8.4)

Under what conditions will a stationary distribution exist? Is it unique?

def: Given a MC $\{X_n : n \geq 0\}$ on a state space S & given a state $i \in S$, the mean return time to state i is

$$\boxed{m_i = E_i(\inf\{n \geq 1 : X_n = i\})}.$$

We always have $m_i \geq 1 \quad \forall i \in S$.

If state i is transient, then $m_i = \infty$.

If state i is recurrent, then either

$m_i = \infty \quad \leftarrow \quad i$ is null recurrent OR

$m_i < \infty \quad \leftarrow \quad i$ is positive recurrent.

* Thm 8.4.1: If a Markov chain is irreducible, and if each state is positive recurrent with finite mean return time m_i, then the MC has a unique stationary distribution $\pi = \{\pi_i\}$ given by

$$\boxed{\pi_i = \frac{1}{m_i}} \text{ for each state } i \in S.$$

Proof of this theorem relies heavily on the following lemma.

Lemma 8.4.2: Let $G_n(i,j) = E_i[\#\{\ell : 1 \le \ell \le n, X_\ell = j\}]$ $= \sum_{\ell=1}^{n} p_{ij}^{(\ell)}$. Then for an irreducible recurrent Markov chain, $\lim_{n\to\infty} \frac{G_n(i,j)}{n} = \frac{1}{m_j}$, for any states i & j.

Pf: Let T_j^r = time of r^{th} hit of state j. Then

$$T_j^r = \underbrace{T_j^1 + (T_j^2 - T_j^1) + \cdots + (T_j^r - T_j^{r-1})}_{\substack{\text{i.i.d.} \\ \text{with mean } m_j}}$$

By SLLNs (2nd version), we have that $\lim_{r\to\infty} \frac{T_j^r}{r} = m_j$ w/prob. 1. (convergence (a.s.))

For $n \in \mathbb{N}$, let $r(n) = \#\{\ell : 1 \le \ell \le n, X_\ell = j\}$.

Then $\lim_{n\to\infty} r(n) = \infty$ with prob. 1 by recurrence. Also,

$$T_j^{r(n)} \le n < T_j^{r(n)+1} \quad \text{\& now divide by } r(n)$$

$$\Rightarrow \frac{T_j^{r(n)}}{r(n)} \le \frac{n}{r(n)} < \frac{T_j^{r(n)+1}}{r(n)} \quad \text{\& take } \lim_{n\to\infty}$$

$$\Rightarrow \lim_{n\to\infty} \frac{n}{r(n)} = m_j \quad \text{with prob. 1}$$

$$\Rightarrow \lim_{n\to\infty} \frac{r(n)}{n} = \frac{1}{m_j} \quad \text{w/prob. } 1$$

On the other hand, $G_n(i,j) = E_i[r(n)]$ and clearly

$0 \le \frac{r(n)}{n} \le 1$. Then by the Bounded Convergence Thm,
(Thm 7.3.1)

$$\lim_{n\to\infty} \frac{G_n(i,j)}{n} = \lim_{n\to\infty} \frac{E_i[r(n)]}{n} = \lim_{n\to\infty} E_i\left[\frac{r(n)}{n}\right]$$

$$= \frac{1}{m_j}. \quad \blacksquare$$

$$\left(E_i\left[\lim_{n\to\infty} \frac{r(n)}{n}\right] = \frac{1}{m_j} \right)$$

Pf of Thm 8.4.1:

[Exercise for the reader!]

Prove uniqueness: $\pi_j = \frac{1}{m_j}$ is the only possible stat. dist'n.

Show that $C = \sum_j \frac{1}{m_j} = 1$.

Note: States that are not positive recurrent cannot contribute to a stationary dist'n.

Prop 8.4.5: If a MC has a stationary distribution $\pi = \{\pi_i\}$ and if a state j is not positive recurrent (i.e. $m_j = \infty$), then $\pi_j = 0$.

Cor 8.4.6 : If a MC has no positive recurrent states, then it does not have a stationary dist'n.

(all $\pi_j = 0$ which contradicts $\sum_j \pi_j = 1$)

Cor 8.4.7 : Let i & j be 2 states of a MC that communicate (i.e. $f_{ij} > 0$ and $f_{ji} > 0$). If i is positive recurrent then so is j.

Cor 8.4.8 : For an irreducible MC (i.e. all states communicate), either all states are positive recurrent or none are.

* Thm 8.4.9 : For an irreducible MC, either

(a) All states are positive recurrent, & $\exists$ a unique stationary dist'n given by $\pi_j = \frac{1}{m_j}$

(if also aperiodic, $P_i(X_n = j) \to \pi_j$ as $n \to \infty$), OR

(b) No states are pos. recurrent, and $\nexists$ a stationary distribution.

Example 1 : Symmetric simple RW on $\mathbb{Z}$ is null recurrent, falls into category (b).

Example 2 : MC on finite S necessarily falls into category (a)..

Prop 8.4.10: For an irreducible MC on a finite state space, all states are positive recurrent (& hence a uniqe stat. dist'n exists).

skip in class

PF: Fix state $i \in S$. Let $h_{ji}^{(m)} = P_j(X_k = i \text{ for some } 1 \leq k \leq m)$

$$= \sum_{n=1}^{m} f_{ji}^{(n)}.$$

Then $\lim_{m\to\infty} h_{ji}^{(m)} = \lim_{m\to\infty} \sum_{n=1}^{m} f_{ji}^{(n)} = f_{ji}$ (by def)

> 0 by irreducibility. (for each $j \in S$)

Since S is finite, we can find $m \in \mathbb{N}$ and $\delta > 0$ s.t. $h_{ji}^{(m)} \geq \delta \quad \forall j$.

We must also have $1 - h_{ii}^{(n)} \leq (1-\delta)^{\lfloor n/m \rfloor}$ so that letting $\tau_i = \inf\{n \geq 1 : X_n = i\}$, we have that

$$m_i = \sum_{n=0}^{\infty} P_i(\tau_i \geq n+1) = \sum_{n=0}^{\infty} (1 - h_{ii}^{(n)})$$

$$\leq \sum_{n=0}^{\infty} (1-\delta)^{\lfloor n/m \rfloor}$$

$$= \frac{m}{\delta} < \infty.$$

Thus, all states are pos. recurrent.

Limit Theorems

(Ref: R section 9, B section 16)

Some results needed for more advanced topics to come.

Suppose $X_1, X_2, \ldots$ are random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Also, suppose that

$$\lim_{n\to\infty} X_n(\omega) = X(\omega)$$ for all $\omega \in \Omega$ outside a set of probability 0.

(for each fixed ω)

In other words,

$$X_n \xrightarrow{a.s.} X$$

Q. Does it follow that $\lim_{n\to\infty} E[X_n] = E[X]$?

No, not true in general.

Simple counter example:

$\Omega = \mathbb{N}$

$P(\omega) = 2^{-\omega}$

$$X_n(\omega) = \begin{cases} 2^n & \text{if } \omega = n \\ 0 & \text{if } \omega \neq n \end{cases}$$

(also denoted $X_n(\omega) = 2^n \delta_{\omega,n}$)

Then $X_n \to 0$ with prob. 1 but

$$E[X_n] = 2^n \cdot 2^{-n} + 0 \cdot 2^{-\omega} = 1 \not\to 0 \text{ as } n \to \infty$$