$|ab| = |a||b|$

$$\leq \int \left| e^{i(t+h)x} - e^{itx} \right| d\mu(x)$$

$$= \int \underbrace{|e^{itx}|}_{=1} |e^{ihx} - 1| \, d\mu(x) = \int |e^{ihx} - 1| \, d\mu(x)$$

Lec 22

1a As $h \to 0$, this decreases to 0 by the Bounded Convergence Thm (since $|e^{ihx} - 1| \leq 2$). Hence, $\phi_X(t)$ is always a uniformly continuous function.

start here

Derivatives of $\phi_X(t)$ are straightforward.

Prop 11.0.1: Suppose X is a RV with $E[|X|^k] < \infty$. Then for $0 \leq j \leq k$, ϕ_X has finite j^{th} derivative given by

$$\boxed{\phi_X^{(j)}(t) = E[(iX)^j e^{itX}]}.$$

In particular, evaluate at $t = 0$ to get

$$\boxed{\phi_X^{(j)}(0) = i^j E[X^j]}.$$

Pf: Uses induction on j. See text for details.

The Continuity Theorem

Main Idea: Characteristic functions can be used to study limit distributions.

First, an inversion theorem – tells how to recover info about a probability distn from its char. function.

Thm 11.1.1 [Fourier Inversion Thm]: Let μ be a Borel probability measure on $\mathbb{R}$ with characteristic function $\phi(t) = \int_{\mathbb{R}} e^{itx}\, d\mu(x)$. Then if $a<b$ and $\mu(\{a\}) = \mu(\{b\}) = 0$ then

$$\mu([a,b]) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\, \phi(t)\, dt$$

Pf requires 2 lemmas:

Lemma 1: For $T \geq 0$ and $a<b$,

$$\int_{\mathbb{R}} \int_{-T}^{T} \left| \frac{e^{-ita} - e^{-itb}}{it}\, e^{itx} \right| dt\, d\mu(x) \leq 2T(b-a) < \infty$$

Lemma 2: For $T \geq 0$ and $\theta \in \mathbb{R}$,

$$\lim_{T\to\infty} \int_{-T}^{T} \frac{\sin(\theta t)}{t}\, dt = \pi\, \mathrm{sgn}(\theta),$$

where $\mathrm{sgn}(\theta) = \begin{cases} 1 & \text{if } \theta > 0 \\ -1 & \text{if } \theta < 0 \\ 0 & \text{if } \theta = 0 \end{cases}$

Furthermore, $\exists\, M < \infty$ s.t. $\left| \int_{-T}^{T} \frac{\sin(\theta t)}{t}\, dt \right| \leq M \quad \forall\, T \geq 0 \text{ \& } \theta \in \mathbb{R}$.

Pf of Thm 11.1.1 : Compute that

$$\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-ita}-e^{-itb}}{it}\phi(t)\,dt = \frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-ita}-e^{-itb}}{it}\underbrace{\left(\int_{\mathbb{R}} e^{itx}\,d\mu(x)\right)}_{\text{by def'n of } \phi(t)} dt$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{-T}^{T}\frac{e^{it(x-a)}-e^{it(x-b)}}{it}\,dt\,d\mu(x)$$ by Fubini's Thm & Lemma 1 (need finite $\int$ to use)

$$= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{-T}^{T}\frac{\sin(t(x-a))-\sin(t(x-b))}{t}\,dt\,d\mu(x)$$

(note that $\frac{\cos(t(x-a))-\cos(t(x-b))}{it}$ is an odd function in t)

Now use Lemma 2 along with Bounded Convergence Thm to show that

$$\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-ita}-e^{-itb}}{it}\phi(t)\,dt$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}}\pi\left(\operatorname{sgn}(x-a)-\operatorname{sgn}(x-b)\right)d\mu(x)$$

$$= \int_{-\infty}^{\infty}\underbrace{\frac{1}{2}\left(\operatorname{sgn}(x-a)-\operatorname{sgn}(x-b)\right)}\,d\mu(x)$$

Recall that $\operatorname{sgn}(x-s) = \begin{cases} 1 & \text{if } x>s \\ -1 & \text{if } x<s \\ 0 & \text{if } x=s \end{cases}$

& that $a<b$

Then $\frac{1}{2}\left(\operatorname{sgn}(x-a) - \operatorname{sgn}(x-b)\right)$ is $\frac{1}{2}\cdot 0 \begin{cases} \text{if } x<a \ (<b) \\ \text{OR} \\ x>b \ (>a) \end{cases}$

OR

$\frac{1}{2}\cdot 1 \begin{cases} \text{if } x=a<b \\ \text{OR } x=b \ (x=b>a) \end{cases}$

OR

$\frac{1}{2}\cdot 2 \ \{ \text{if } a<x<b$

Continuing from previous eqns,

$$= \frac{1}{2}\mu(\{a\}) + \mu\left((a,b)\right) + \frac{1}{2}\mu(\{b\}).$$

But if $\mu(\{a\}) = \mu(\{b\}) = 0$ as assumed, then this is $\mu([a,b])$, as claimed. $\boxtimes$

Cor 11.1.7 [Fourier Uniqueness Thm]: Let X & Y be RVs.

Then $\phi_X(t) = \phi_Y(t)$ $\forall t \in \mathbb{R}$ iff $\mathcal{L}(X) = \mathcal{L}(Y)$.

(i.e. iff X & Y have the same distribution).

Pf: ($\Longrightarrow$) Suppose $\phi_X(t) = \phi_Y(t)$ $\forall t \in \mathbb{R}$. Then (by Inversion Thm)

$P(a \le X \le b) = P(a \le Y \le b)$ provided that

$P(X=a) = P(X=b) = P(Y=a) = P(Y=b) = 0$,

i.e. for all but countably many choices for a & b.

Now take limits & use continuity of probabilities:

$\Rightarrow P(X \in I) = P(Y \in I)$ for all intervals $I \subseteq \mathbb{R}$.

It follows from Prop 2.5.8 (uniqueness of extensions of prob. meas.)

that $\mathcal{L}(X) = \mathcal{L}(Y)$.

($\Longleftarrow$) Conversely, suppose $\mathcal{L}(X) = \mathcal{L}(Y)$. Then

$E[e^{itX}] = E[e^{itY}]$ by Cor 6.1.3.

$E[f(X)] = E[f(Y)]$ $\forall$ Borel-meas $f: \mathbb{R} \to \mathbb{R}$ s.t. exp. value is well-defined.

$\Rightarrow \phi_X(t) = \phi_Y(t)$ $\forall t \in \mathbb{R}$

by definition.

Need further results in order to prove the continuity theorem.

Lemma 11.1.8 [Helly Selection Principle]: Let $\{F_n\}$ be a sequence of CDFs ($F_n(x) = \mu_n((-\infty, x])$) for some probability distribution μ_n. Then there is a subsequence $\{F_{n_k}\}$ and a non-decreasing right-continuous function F with $0 \le F \le 1$ s.t. $\lim_{k\to\infty} F_{n_k}(x) = F(x) \ \forall x \in \mathbb{R}$ s.t. F is continuous at x.

[Pf: SKIP]

Note: This lemma does ensure that

$$\lim_{x\to\infty} F(x) = 1 \quad \text{or} \quad \lim_{x\to-\infty} F(x) = 0$$

(as we'd expect from a CDF)

To get around this, define a new term: tight

def: A collection of $\{\mu_n\}$ probability measures on $\mathbb{R}$ is ~~tight if $\forall \varepsilon > 0$, $\exists\ a<b$ with $\mu_n([a,b]) \ge 1-\varepsilon \ \forall n$.~~

In words, all of the measures have "most" ($1-\varepsilon$) of their mass attributed to the same finite interval $[a,b]$. Mass does not escape off to ∞.

Example : Let $\mu_n = \delta_{n \bmod 3}$ — point mass at $n \bmod 3$.

i.e. $\mu_1 = \delta_1$, $\mu_2 = \delta_2$, $\mu_3 = \delta_0$ } repeats

$\mu_4 = \delta_1$, $\mu_5 = \delta_2$, $\mu_6 = \delta_0$ }

$\vdots$

Q. Is $\{\mu_n\}$ tight?

Yes. Take $[a,b] = [0,3]$.

Then $\forall \varepsilon > 0$ & $\forall n \in \mathbb{N}$,

$$\mu_n([0,3]) = \delta_n([0,3])_{\bmod 3}$$

$$= \mathbb{1}_{[0,3]}(n)\Big|_{n=0,1,2}$$

$$= 1 > 1-\varepsilon.$$

Example : For $n \geq 1$, let $X_n \sim \text{Uniform}(a_n, 2+a_n)$

where $a_n = (-1)^n$. Then

$X_1 = U(-1, 2-1=1)$ so $|X_1| \leq 1$

$X_2 = U(1, 2+1=3)$ so $|X_2| \leq 3$

$X_3 = U(-1, 1)$

$X_4 = U(1,3)$

$\vdots$

$\Rightarrow |X_n| \leq 3 \quad \forall n \in \mathbb{N}$.

$\Rightarrow \sup_{n \geq 1} P(|X_n| > 3) < \varepsilon \quad \forall \varepsilon > 0$

$\Rightarrow$ seq. of prob. dist'ns of $\{X_n\}$ is tight