Conditional Probability & Expectation

(Ref: § 13 Rosenthal, § 33-34 Billingsley)

- Conditioning on events of positive measure – straightforward

A & B events with $P(B) > 0$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$ – defn of conditional probability

More generally, let Y be a RV and define ν by

$$\nu(S) = P(Y \in S | B) = \frac{P(Y \in S, B)}{P(B)}$$

Then $\nu = \mathcal{L}(Y|B)$ is a probability measure called the conditional distribution of Y given B

Conditional Expectation: $E[Y|B] = \int y \, d\nu(y)$

Also, $\mathcal{L}(Y \mathbb{1}_B) = P(B)\mathcal{L}(Y|B) + P(B^c)\delta_0$ so taking expectations & rearranging yields

$$\Rightarrow \quad E[Y|B] = \frac{E[Y \mathbb{1}_B]}{P(B)}$$

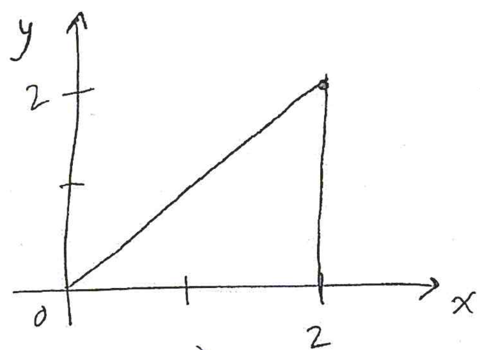
- Conditioning on events of measure 0 (i.e. $P(B)=0$) is not as straightforward, frequently arises.

This approach does not work.

Conditioning on a random variable

Example : (X,Y) uniformly distributed on triangle

$$T = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 2,\ y \le x \le 2\}$$



where

$P((X,Y)\in S) = \frac{1}{2}\lambda_2(S \cap T)$

for Borel set $S \subseteq \mathbb{R}^2$

& λ_2 = Lebesgue meas. on $\mathbb{R}^2$

Q. What is $P(Y \ge \frac{3}{4} \mid X=1)$?

$E[Y \mid X=1]$?

Not clear how to proceed since $P(X=1)=0$ ...

we will come back to this soon!

Different Approach:

Given a RV X, consider probabilities such as $P(A \mid X)$ and cond. expected values $E[Y \mid X]$

to be themselves RANDOM VARIABLES !

- Think of these as functions of the random value X
- Counter-intuitive: $P(\cdots)$ & $E(\cdots)$ are usually numbers, not RVs
- This idea allows us to partially resolve the issue of conditioning on sets of measure 0
- Require that these RVs have the correct exp. values:

$$\left.\begin{aligned} E[P(A|X)] &= P(A) \\ E[E[Y|X]] &= E[Y] \end{aligned}\right\} (*)$$

- Need a stronger requirement (correct mean isn't enough) → lots of dist'ns have same mean!

<u>def</u>: Given RVs X & Y with $E[|Y|] < \infty$ and an event A, $P(A|X)$ is a <u>conditional probability</u> of A given X if it is a $\sigma(X)$-measurable RV, and for any Borel set $B \subseteq \mathbb{R}$,

$$\underbrace{E[P(A|X)\, \mathbb{1}_{X\in B}]}_{= \int_B P(A|X)\, dP} = P(A \cap \{X \in B\}).$$

def: $E[Y|X]$ is a conditional expectation of Y given X if it is a $\sigma(X)$-measurable RV, and $\forall B \in \mathcal{B}$ (Borel σ-algebra)

$$E\left[E[Y|X] \mathbb{1}_{X \in B} \right] = E\left[Y \mathbb{1}_{X\in B} \right]$$

OR $\left(\int_B E[Y|X]\, dP = \int_B Y\, dP \right)$

Recall: Prob. Space $(\Omega, \mathcal{F}, P)$, $\mathcal{G}$ is a sub-σ-algebra (i.e. a σ-algebra contained in $\mathcal{F}$), then a RV Z is $\mathcal{G}$-measurable if $\{Z \le z\} \in \mathcal{G}$ $\forall z \in \mathbb{R}$.

e.g. $\sigma(X) = \{ \{X \in B\} : B \in \mathcal{B} \}$ (Borel σ-algebra)

Remarks:

① Requiring these quantities to be $\sigma(X)$-measurable means they are functions of X alone, don't o.w. depend on the sample point $\omega \in \Omega$.

If RV Z is $\sigma(X)$-meas., then for each $z \in \mathbb{R}$,

$\{Z = z\} = \{X \in B_z\}$ for some Borel set $B_z \subseteq \mathbb{R}$.

Then Z is a function of X:

$Z = f(X)$ where f is s.t. $f(x) = z$ $\forall x \in B_z$

② If we set $B = \mathbb{R}$ in the above defns, then we get the special case (*).

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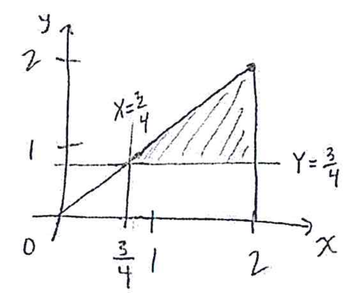
3. Exp. values are unaffected by changes on a set of measure 0, so cond. prob. & expectations are only unique up to a set of meas. 0.

4. These conditional probabilities & expectations always exist: $P(A \mid X)$ and $E[Y \mid X]$.

Back to motivating example (triangle):



$$P\left(Y \geq \tfrac{3}{4} \mid X\right) = \begin{cases} \dfrac{X - \frac{3}{4}}{X}, & X \geq \frac{3}{4} \\ 0, & X < \frac{3}{4} \end{cases}$$

$$E[Y \mid X] = \frac{X}{2}$$

We can then compute

(def 1)

"and", $\cap$

$E\left[P\left(Y \geq \frac{3}{4} \mid X\right) \mathbb{1}_{X \in B}\right]$ & show it equals $P\left(Y \geq \frac{3}{4}, X \in B\right)$.

Likewise compute

(def 2)

$E\left[E[Y \mid X] \mathbb{1}_{X \in B}\right]$ & show that it equals $E\left[Y \mathbb{1}_{X \in B}\right]$.