Martingales (con't)

Recall Def: A stochastic process $\{X_n : n = 0,1,2,\ldots\}$ is a martingale if $E[|X_n|] < \infty$ $\forall n$ and w.p. 1

$$E[X_{n+1} \mid \underbrace{X_0, X_1, \ldots, X_n}_{\mathcal{F}_n = \sigma(X_0,\ldots,X_n)}] = X_n$$

Example 1: Unbiased random walk on $\mathbb{Z}$ where $X_0 = 0$,

$p_{i,i+1} = p_{i,i-1} = \frac{1}{2}$ $\forall i \in \mathbb{Z}$. (Actually true in any dimension $\mathbb{Z}^d$)

Example 2: Let $Z_1, Z_2, \ldots$ be independent RVs. Let

$X_0 = 0$ & $X_n = Z_1 + \cdots + Z_n$ for $n = 1,2,\ldots$ (OR $X_n = \sum_{i=1}^n Z_i$)

Then $X_{n+1} = X_n + Z_{n+1}$.

$$E[X_{n+1} \mid X_0,\ldots,X_n] = E[X_n \mid X_0,\ldots,X_n] + E[Z_{n+1} \mid X_0,\ldots,X_n]$$

$$= X_n + E[Z_{n+1}]$$

since $X_n \in \sigma(X_0,\ldots,X_n)$

since Z_{n+1} is indep of $X_0,\ldots,X_n$

Thus, $\{X_n\}$ is a martingale iff

$$E[Z_1] = E[Z_2] = \cdots = 0.$$

Example 3: Let $Z_1, Z_2, \ldots$ be indep. RVs. Let $X_n = \sum_{i=1}^{n} Z_n$

& $Y_n = e^{X_n}$. Then $Y_{n+1} = e^{X_{n+1}} = e^{X_n + Z_{n+1}} = Y_n e^{Z_{n+1}}$.

$$E[Y_{n+1} \mid Y_0, \ldots, Y_n] = E[Y_n e^{Z_{n+1}} \mid Y_0, \ldots, Y_n]$$

$$= Y_n E[e^{Z_{n+1}} \mid Y_0, \ldots, Y_n]$$

$$= Y_n E[e^{Z_{n+1}}]$$ since Z_{n+1} is indep of $Y_0, \ldots, Y_n$

Thus, $\{Y_n\}$ is a martingale iff

$$E[e^{Z_1}] = E[e^{Z_2}] = \cdots = 1.$$

def: A stochastic process $\{X_n\}$ is a submartingale if $E[|X_n|] < \infty \ \forall n$ and

$$\boxed{E[X_{n+1} \mid X_0, \ldots, X_n] \geq X_n}.$$

def: Stoch. process $\{X_n\}$ is a supermartingale if $E[|X_n|] < \infty \ \forall n$ and

$$\boxed{E[X_{n+1} \mid X_0, \ldots, X_n] \leq X_n}.$$

* These names are arguably the reverse of what they should be intuitively.

* A martingale $\{X_n\}$ can be thought of as the fortune at time n of a player who is betting on a fair game.
  - Submartingale – as outcome of betting on a favorable game
  - Supermartingale – as outcome of betting on an unfavorable game

Result 1: You cannot make money betting on martingales! In particular, if you choose to stop playing at some bounded time N, then expected winnings $E[X_N] = E[X_0]$.
initial fortune

To explain this, suppose $\{X_n\}$ is a submartingale.

$$E[X_{n+1} \mid X_0, \dots, X_n] \geq X_n$$

$$\Rightarrow E\big[E[X_{n+1} \mid X_0, \dots, X_n]\big] \geq E[X_n]$$

$$\Rightarrow E[X_{n+1}] \geq E[X_n] \text{ \& by induction}$$

$$E[X_n] \geq E[X_{n-1}] \geq \dots \geq E[X_1] \geq E[X_0]$$

So if $\{X_n\}$ is a martingale, then

$$E[X_n] = E[X_{n-1}] = \cdots = E[X_0] \quad \text{or} \quad E[X_n] = E[X_0] \quad \forall n \in \mathbb{N}.$$

Q. What about $E[X_\tau]$ where τ is a <u>random time</u>?

Is it still true that $E[X_\tau] = E[X_0]$?

<u>Ans:</u> No, in general.

RV $\tau(\omega)$ so really have $X_{\tau(\omega)}(\omega)$

<u>def</u>: A random variable τ (which takes on values $0, 1, 2, \ldots$) is called a <u>stopping time</u> if the event $\{\tau = n\}$ only depends on $X_0, \ldots, X_n$ and not on any future information (say X_{n+1} for instance).

hitting time problems

<u>Example 1</u>: $\tau = \min\{n : X_n > 1\}$ is a stopping time.

$\{\tau = n\}$ means that $X_0 \le 1, X_1 \le 1, \ldots, X_{n-1} \le 1$ and $X_n > 1$, which only depends on $X_0, \ldots, X_n$.

<u>Example 2</u>: $\tau = \max\{n : X_n > 1\}$ is NOT a stopping time. Consider $\{\tau = 1\}$. This means that $X_1 > 1$ and $X_2 \le 1$, but this depends on future values of X_n.

last exit problems (last time a process hits a state or set of states)

Thm [Optional Stopping Theorem]: Consider a stopping time τ that is bounded above by a constant N s.t. $\tau \le N$. If $\{X_n\}$ is a martingale, then

$$E[X_\tau] = E[X_N] = E[X_0].$$

Martingale Convergence

If $\{X_n\}$ is a martingale (or submartingale), will it converge a.s. to some random value?

Ans: In general, no.

e.g. Simple symmetric RW is a martingale, but it is null recurrent & has no stationary dist'n.

Result 2: Concerns submartingales (stochastic analogues of non-decreasing sequences): If they are bounded above, they converge to a limit a.s.

↳ Known as the Martingale Convergence Thm

Thm [Martingale Convergence]: Let $\{X_n\}$ be a submartingale. Suppose that $\sup_n E[|X_n|] < \infty$. Then there is a finite RV X s.t.

$$X_n \xrightarrow{a.s.} X.$$

Example: Markov chain $\{X_n\}$ on non-negative integers,

$X_0 = 50$, transition probabilities $P_{ij} = \frac{1}{2i+1}$, $0 \le j \le 2i$

and $P_{ij} = 0$ o.w.

That is, if $X_n = i$ then $X_{n+1} \sim$ Uniform on $\{0, 1, \ldots, 2i\}$

$$\begin{cases} X_0 = 50 = i \\ P_{50,j} = \frac{1}{2(50)+1} = \frac{1}{101} \quad \text{for } 0 \le j \le 100 \end{cases}$$ & so on.

This MC is a Martingale (exercise to reader to check!) & hence also a submartingale

It follows that $X_n \xrightarrow{a.s.} 0$.