Probability Measures (§2 & Rosenthal Ch. 2)

def: A probability triple or probability measure space is given by $(\Omega, \mathcal{F}, P)$ where

- the sample space Ω is any non-empty set
- the σ-algebra or σ-field $\mathcal{F}$ is a collection of subsets of Ω:

  containing Ω itself & the empty set ϕ,
  closed under complements, countable unions, and countable intersections.
- the probability measure P is a mapping (function) from $\mathcal{F}$ to $[0,1]$, with $P(\Omega) = 1$, $P(\phi) = 0$, such that P is countably additive. Also, $0 \le P(A) \le 1 \ \forall A \in \mathcal{F}$.

Example: Uniform distribution on $[0,1]$

$\Omega = [0,1]$

$\mathcal{F}$ contains all intervals $[a,b]$ s.t. $0 \le a \le b \le 1$
but certainly contains many more subsets as well

$P : \mathcal{F} \to [0,1]$ (satisfying properties above)

More Details

- The sample space Ω consists of all possible outcomes ω of an experiment or observation.

  e.g. Count # of heads in n tosses of a coin.

  $\Omega = \{0, 1, 2, \ldots, n\}$

- A subset of Ω is an event and an element ω of Ω is a sample point.

- The σ-algebra $\mathcal{F}$ is a collection (i.e. set) of all events or measurable sets. These are the subsets $A \subseteq \Omega$ for which $P(A)$ is well-defined.

  * We know from the Proposition last time (see Lec 1 notes) that in general $\mathcal{F}$ might not contain
  * ALL subsets of Ω,

  though we still expect it to contain most of the subsets that come up naturally.

def: A class $\mathcal{F}$ of subsets of Ω is a σ-field if

(i) $\Omega \in \mathcal{F}$

(ii) For any subset $A \subseteq \Omega$, if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

(iii) For any countable (or finite) collection of subsets $A_1, A_2, A_3, \ldots \subseteq \Omega$,

- if $A_i \in \mathcal{F}$ for each i, then $A_1 \cup A_2 \cup A_3 \cup \cdots \in \mathcal{F}$

  (short-hand: $A_i \in \mathcal{F} \ \forall i \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$)

- if $A_i \in \mathcal{F}$ for each i, then $A_1 \cap A_2 \cap A_3 \cap \cdots \in \mathcal{F}$

  ($A_i \in \mathcal{F} \ \forall i \Rightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$)

Remark: As we did for countable additivity, we require $\mathcal{F}$ to be closed under countable operations to allow for taking limits when studying prob. theory.

Also cannot extend the above def'n to uncountable case!

Q. Smallest σ-field? contains Ω & ϕ, nothing else

Largest σ-field? power set 2^{Ω} = all possible subsets of Ω

Monotonicity Property

If A & $B \in \mathcal{F}$ and $A \subseteq B$, then $\boxed{P(A) \le P(B)}$.

Principle of Inclusion - Exclusion

If $A, B \in \mathcal{F}$, then

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

Countable subadditivity

For any sequence $A_1, A_2, \ldots \in \mathcal{F}$ (disjoint or not),

$$\boxed{P(A_1 \cup A_2 \cup \cdots) \le P(A_1) + P(A_2) + \cdots}$$

Comes from:

disjoint union

$$\begin{aligned} P(A_1 \cup A_2 \cup \cdots) &= P(A_1 \,\dot\cup\, (A_2 \setminus A_1) \,\dot\cup\, (A_3 \setminus A_2 \setminus A_1) \cdots) \\ &= P(A_1) + P(A_2 \setminus A_1) + P(A_3 \setminus A_2 \setminus A_1) + \cdots \\ &\le P(A_1) + P(A_2) + P(A_3) + \cdots \end{aligned}$$

def: A class $\mathcal{F}$ of subsets of Ω is a field if

(i) $\Omega \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(iii) $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$

(closed under formation of complements & finite unions)

Since $\Omega^c = \phi$

(ii) $\Rightarrow \phi \in \mathcal{F}$

By De Morgan's law,
$$A \cap B = (A^c \cup B^c)^c$$
$$A \cup B = (A^c \cap B^c)^c$$

So if $\mathcal{F}$ is closed under finite unions, then $\mathcal{F}$ is closed under finite intersections

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* $\mathcal{F}$ is a σ-field if it is also closed under the formation of countable unions (& hence countable $\cap$'s)

Constructing Probability Triples

Consider a simple example : Poisson ($\lambda=5$) distribution

$\Omega = \{0,1,2,\ldots\}$ - non-neg. integers

$\mathcal{F}$ consists of all subsets of Ω (i.e. the power class 2^{Ω})

P is defined for any $A \in \mathcal{F}$ by

$$P(A) = \sum_{k \in A} e^{-5} 5^k / k!$$

Check that $\mathcal{F}$ is indeed a σ-algebra :

Since $\mathcal{F}$ contains all subsets of Ω, clearly $\Omega \in \mathcal{F}$, $\phi \in \mathcal{F}$, and it is closed under any set operation. ✓

Check that P is a probability measure defined on $\mathcal{F}$:

$P(\phi) = 0$, $P(\Omega) = 1$, $0 \leq P(A) \leq 1$ for any $A \in \mathcal{F}$,

Countable additivy follows since

if A & $B \in \mathcal{F}$ are disjoint, then

$\sum_{k \in A \cup B} (\cdot)$ is the same as $\sum_{k \in A} (\cdot) + \sum_{k \in B} (\cdot)$. ✓

Construction of an appropriate prob. triple is straightforward for the Poisson (5) distribution.

Similarly straightforward for any discrete probability space.

Ω finite or countable

Thm: Let Ω be a finite or countable non-empty set. Let $p : \Omega \to [0,1]$ be a function satisfying $\sum_{\omega \in \Omega} p(\omega) = 1$. Then there is a valid probability triple $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}$ is the collection of all subsets of Ω, and for $A \in \mathcal{F}$, $P(A) = \sum_{\omega \in A} p(\omega)$.

Example 1: Rolling a fair die

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$\mathcal{F} = 2^{\Omega}$

cardinality of event A

$$P(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{6} \quad \text{for any } A \in \mathcal{F}$$

Example 2: Flipping a (possibly biased) coin

$\Omega = \{H, T\}$

$\mathcal{F} = 2^{\Omega} = \{\phi, \{H\}, \{T\}, \{H, T\}\}$

P satisfies $P(H) = p$, $P(T) = 1-p$ for some $p \in (0,1)$

If the sample space Ω is NOT countable, then the situation is considerably more complex.

Q. How to define $(\Omega, \mathcal{F}, P)$ which corresponds to Uniform $(0,1)$ distribution?

Clearly we want $\Omega = [0,1]$.

What about $\mathcal{F}$?

Proposition (last time) $\Rightarrow \mathcal{F} \neq 2^{\Omega}$ – cannot contain all possible subsets of $\Omega = [0,1]$.

Should contain all intervals:

$[a,b]$, $(a,b]$, $[a,b)$, (a,b) s.t. $0 \leq a \leq b \leq 1$

(i.e. open, closed, half-open, singleton, empty intervals)

So, we must have

$\mathcal{F} \supseteq \mathcal{J}$ where $\mathcal{J} = \{$all intervals contained in $[0,1]\}$

this collection is a semialgebra of subsets of Ω

i.e. contains $\emptyset$ & Ω, closed under finite $\cap$, complement of $A \in \mathcal{J}$ equals a finite disjoint $\cup$ of elements

Q. Since $\mathcal{J}$ is only a semialgebra, how do we create a σ-algebra?

Consider $B_0 = \{$all finite unions of elements of $\mathcal{J}\}$

or $B_1 = \{$all finite or countable unions of elements of $\mathcal{J}\}$

Unfortunately, neither B_0 nor B_1 is a σ-algebra. (HW problem to show this)

Hence, the construction of $\mathcal{F}$ and P presents some challenges.

→ Need a general result (Theorem) about constructing probability spaces when Ω is uncountable.

→ The Extension Theorem