The Extension Theorem

(Ref: R § 2.3, B § 3)

Last time: Probability Measure Spaces
aka Probability Triples

Ω countable:
$(\Omega, \mathcal{F}, P)$ straightforward
$\mathcal{F} =$ Power Set

e.g. $\Omega = \{1, 2, 3\}$

Ω uncountable:
$(\Omega, \mathcal{F}, P)$ complicated
$\mathcal{F}$: NOT the Power set

e.g. $\Omega = [0,1]$

$\boxed{\text{Big Picture via } U(0,1) \text{ example}}$

The Extension Thm allows us to automatically construct valid probability triples for $\Omega = [0,1]$.

* Once a prob. measure is constructed on a semialgebra $(\mathcal{J})$, then it can be extended to a σ-algebra $(\mathcal{M})$.

Recall def:

$\mathcal{J} = \{\text{all intervals contained in } [0,1]\}$

all open, closed, half open, singleton intervals $\subseteq [0,1]$ & ϕ

Q. How to create a σ-algebra?

Recall that

$B_0 = \{\text{all finite unions of elements of } J\}$

$B_1 = \{\text{all finite or countable unions of elements of } J\}$

both are not σ-algebras

What we want is:

- $\mathcal{B} = \sigma(J)$ = Borel σ-algebra

  the σ-algebra generated by J (the set of all open (or closed) intervals in $[0,1]$).

- $\mathcal{M}$ = Lebesgue measurable sets, also a σ-algebra.

Note that $\boxed{\mathcal{B} \subseteq \mathcal{M}}$.

Also note that $\mathcal{B}$ is generally not equal to 2^{Ω} (power set).

Example of a non-Borel set: Vitali Set
(non-measurable set)

$\mathcal{M}$ contains more sets than $\mathcal{B}$ (generally)

$\mathcal{M}$ is a complete measure space, $\mathcal{B}$ is not complete.

i.e. every subset of a null set is measurable (having meas. 0)

Thm 2.3.1 [The Extension Thm] : Let $\mathcal{J}$ be a semialgebra of subsets of Ω. Let $P: \mathcal{J} \to [0,1]$ with $P(\phi)=0$ and $P(\Omega)=1$, satisfying finite superadditivity :

$$P\left(\bigcup_{i=1}^{k} A_i\right) \geq \sum_{i=1}^{k} P(A_i) \quad \text{whenever } A_1, \dots, A_k \in \mathcal{J}, \ \bigcup_{i=1}^{k} A_i \in \mathcal{J}, \text{ and } \{A_i\} \text{ are disjoint,} \quad (*)$$

and also countable monotonicity :

$$P(A) \leq \sum_i P(A_i) \text{ for } A, A_1, A_2, \dots \in \mathcal{J} \text{ with } A \subseteq \bigcup_i A_i . \quad (**)$$

Then there is a σ-algebra $\mathcal{M} \supseteq \mathcal{J}$ and a countably additive probability measure P^* on $\mathcal{M}$ s.t.

$$P^*(A) = P(A) \quad \forall A \in \mathcal{J}.$$

(That is, $(\Omega, \mathcal{M}, P^*)$ is a valid prob. triple which agrees with our previous probabilities on $\mathcal{J}$).

i.e. For an interval $I \in \mathcal{J}$, $P(I)$ = length of interval

$$I = [a,b] \Rightarrow P(I) = b-a$$

Note: Conclusions of this Thm ⇒ (*) must actually hold with equality. However, we only need inequality to apply Thm.

* This theorem provides a way to construct (complicated) prob. triples on a full σ-algebra, using only probabilities defined on the much simpler subsets (e.g. intervals) in $\mathcal{J}$.

Q. How to prove this Thm?

<u>Key Idea</u> : <u>Outer measure</u> P^* defined by

$$\boxed{P^*(A) = \inf_{\substack{A_1, A_2, \ldots \in \mathcal{J} \\ A \subseteq \bigcup_i A_i}} \sum_i P(A_i)} \quad \text{for } A \subseteq \Omega$$

In other words, define $P^*(A)$ for any subset $A \subseteq \Omega$ to be the infimum of sums of $P(A_i)$, where $\{A_i\}$ is any countable collection of elements from the original semialgebra $\mathcal{J}$ whose union contains A.

Use the values of $P(A)$ for $A \in \mathcal{J}$ to help us define $P^*(A)$ for any $A \subseteq \Omega$.

We know that P^* will not necessarily be a proper prob. measure $\forall A \subseteq \Omega$ (recall prop. from Lecture 1)

existence of non measurable sets

However, it is still useful that $P^*(A)$ is at least defined $\forall A \subseteq \Omega$.

* We will show that P^* is a probability measure on some σ-algebra $\mathcal{M}$, and that P^* is an extension of P.

Properties of P^* (outter measure)

- $P^*(\phi) = 0$ (take $A_i = \phi \ \forall i$)
- Monotone Property holds: $A \subseteq B \Rightarrow P^*(A) \leq P^*(B)$

Lemma 1: P^* is an extension of P, i.e.

$$P^*(A) = P(A) \quad \forall A \in \mathcal{J}.$$

Proof: Let $A \in \mathcal{J}$. It follows from (**) in the Extension Thm that $P^*(A) \geq P(A)$ $\left(\text{since } P(A) \leq \sum_i P(A \cap A_i) \leq \sum_i P(A_i) \text{ where } A_i \in \mathcal{J} \text{ and } A \subseteq \bigcup_i A_i\right)$.

On the other hand, choosing $A_1 = A$ and $A_i = \phi$ for $i \geq 2$ in def. of P^*

$\Rightarrow P^*(A) \leq P(A)$. Thus, $P^*(A) = P(A) \ \forall A \in \mathcal{J}$.

Lemma 2: P^* is countably subadditive, i.e.

$$P^*\left(\bigcup_{n=1}^{\infty} B_n\right) \le \sum_{n=1}^{\infty} P^*(B_n) \quad \text{for any } B_1, B_2, \ldots \subseteq \Omega.$$

Pf: See Rosenthal.

Now set

$$\mathcal{M} = \{A \subseteq \Omega : P^*(A \cap E) + P^*(A^c \cap E) = P^*(E) \ \forall\, E \subseteq \Omega\}$$

↑ set of all subsets A with property that P^* is additive on the union of $A \cap E$ with $A^c \cap E$ $\forall\, E \subseteq \Omega$

By subadditivity, we always have

$$P^*(A \cap E) + P^*(A^c \cap E) \ge P^*(E)$$

So the def of $\mathcal{M}$ above is equivalent to

$$\mathcal{M} = \{A \subseteq \Omega : P^*(A \cap E) + P^*(A^c \cap E) \ge P^*(E) \ \forall\, E \subseteq \Omega\}$$

(sometimes helpful!)

Next, we will show that

$\boxed{P^* \text{ is countably additive on } \mathcal{M}.}$

Lemma 3: If $A_1, A_2, \ldots \in \mathcal{M}$ are disjoint, then

$$P^*\left(\bigcup_n A_n\right) = \sum_n P^*(A_n).$$

* Countably *
Additive

Pf: Suppose A_1 & A_2 are disjoint, and $A_1 \in \mathcal{M}$. Then

$$P^*(A_1 \cup A_2) = P^*(A_1 \cap (A_1 \cup A_2)) + P^*(A_1^c \cap (A_1 \cup A_2))$$

since $A_1 \in \mathcal{M}$

$$= P^*(A_1) + P^*(A_2)$$

since A_1 & A_2 are disjoint.

By induction, this lemma holds for any finite set $\{A_i\}$. With countably many disjoint $A_i \in \mathcal{M}$, we see that for any $m \in \mathbb{N}$,

$$\sum_{n \le m} P^*(A_n) = P^*\left(\bigcup_{n \le m} A_n\right) \le P^*\left(\bigcup_n A_n\right).$$

by monotonicity

Since this is true for any $m \in \mathbb{N}$, it follows that

$$\sum_n P^*(A_n) \le P^*\left(\bigcup_n A_n\right).$$

On the other hand, by subadditivity we have

$$\sum_n P^*(A_n) \geq P^*\left(\bigcup_n A_n\right).$$

Thus, lemma holds for countably many A_i as well. ∎

Big Picture once again:

Lemmas 1, 3, 7, 8 (last 2 still to be shown)

$\Rightarrow$ $\mathcal{M}$ is a σ-algebra containing $\mathcal{J}$,

P^* is a probability measure on $\mathcal{M}$, &

P^* is an extension of P.

This proves the Extension Theorem.

Apply the Extension Thm to conclude the following:

Thm 2.4.4: There exists a probability triple $(\Omega, \mathcal{M}, P^*)$ such that $\Omega = [0,1]$, $\mathcal{M}$ contains all intervals in $[0,1]$, and for any interval $I \subseteq [0,1]$, $P^*(I)$ is the length of I.

- The probability measure is Lebesgue measure on $[0,1]$.
- $\mathcal{M}$ is a σ-algebra that contains $\mathcal{B}$, the Borel σ-algebra of subsets of $[0,1]$.

Note: $\mathcal{J} \subseteq \mathcal{B} \subseteq \mathcal{M}$

Moving forward, we will consider $(\underset{\text{uncountable}}{\Omega}, \overset{\text{Borel } \sigma\text{-algebra}}{\mathcal{B}}, \underset{\text{Lebesgue measure on } \mathcal{B}}{P})$