Lecture 5

Stat 706

<u>Note</u>: There also exist <u>uncountable</u> sets with Lebesgue measure 0.

<u>Simplest Example</u>: Cantor Set K

Begin with $[0,1]$. Then remove the open interval $(\frac{1}{3}, \frac{2}{3})$.

Continue removing the open middle intervals of these 2 pieces, etc.

⋮

Continue.

The complement of the Cantor set K^c has Leb. meas. $= 1$

$$\lambda(K^c) = \frac{1}{3} + 2\left(\frac{1}{9}\right) + 4\left(\frac{1}{27}\right) + \cdots$$

$$= \sum_{n=1}^{\infty} 2^{n-1}/3^n = 1.$$

$\frac{1}{2} \cdot 2^n = 2^{n-1}$

$\frac{1}{2} \sum (\frac{2}{3})^n$

Thus, since $P(A) = 1 - P(A^c)$, we have

$$P(K) = 1 - P(K^c) = 1 - 1 = 0.$$

K is uncountable (see justification in Rosenthal).

Extensions of the Extension Thm

Uniqueness Property:

Prop: Let $\mathcal{J}$, P, P^* and $\mathcal{M}$ be as in the extension theorem. Let $\mathcal{F}$ be any σ-algebra with

$$\mathcal{J} \subseteq \mathcal{F} \subseteq \mathcal{M} \quad (\text{e.g. } \mathcal{F} = \mathcal{M} \text{ or } \mathcal{F} = \sigma(\mathcal{J}))$$

σ-algebra generated by semialg. $\mathcal{J}$

Let Q be any probability measure on $\mathcal{F}$ s.t. $Q(A) = P(A) \ \forall A \in \mathcal{J}$.

Then $Q(A) = P^*(A) \ \forall A \in \mathcal{F}$.

Useful special case: $\mathcal{F} = \mathcal{B}$ - Borel subsets of $\mathbb{R}$ (or of $[0,1]$)

Random Variables

(ref §5, Rosenthal §3) – Although, Billingsley focuses on simple RVs, those with finite range

Main Idea: If we think of Ω as the set of all possible random outcomes of an experiment, then a random variable assigns a numerical value to these outcomes.

def: Given a probability triple (prob. space aka) $(\Omega, \mathcal{F}, P)$, a simple random variable is a function X from Ω to $\mathbb{R}$ such that

$$\{\omega \in \Omega : X(\omega) = x\} \in \mathcal{F}, \quad \text{for each } x \in \mathbb{R}.$$

In other words, the function X must be measurable. ($\mathcal{F}$-measurable)

Could also write:

$$\{X = x\} \in \mathcal{F} \quad \forall x \in \mathbb{R}$$

$$X^{-1}(\{x\}) \in \mathcal{F} \quad \forall x \in \mathbb{R}$$

for general RV (not necess. simple)

$$\left[\begin{array}{l} X^{-1}(B) \in \mathcal{F} \text{ for every Borel set } B \\ \{X \in B\} \in \mathcal{F} \text{ for every Borel set } B \end{array}\right]$$

<u>Note</u>: Not all functions from Ω to $\mathbb{R}$ are RVs.

<u>Example</u>: Let $(\Omega, \mathcal{F}, P)$ be Lebesgue measure on $[0,1]$, and let $H \subset \Omega$ be the non-measurable set from Lecture 1 (uses equiv. classes & shift operator).

Prop. 2.4.8 (Rosenth says that $H \notin \mathcal{F}$

Define $X : \Omega \to \mathbb{R}$ by $X = 1_{H^c}$, so

$$\begin{cases} X(\omega) = 0 & \text{for } \omega \in H \\ X(\omega) = 1 & \text{for } \omega \notin H. \end{cases}$$

Then $\{\omega \in \Omega : X \le \frac{1}{2}\} = H$ but $H \notin \mathcal{F}$ so X is not a RV.

<u>Proposition</u>: Given $(\Omega, \mathcal{F}, P)$.

(i) If $X = 1_A$ is the indicator function of some event $A \in \mathcal{F}$, then X is a random variable.

(ii) If X & Y are RVs and $c \in \mathbb{R}$, then

$$\left.\begin{matrix} X + c & X + Y \\ cX & XY \\ X^2 & \end{matrix}\right\} \text{ are all RVs.}$$

(iii) If $Z_1, Z_2, \ldots$ are RVs s.t. $\lim_{n\to\infty} Z_n(\omega)$ exists for each $\omega \in \Omega$ and $Z(\omega) = \lim_{n\to\infty} Z_n(\omega)$, then Z is also a RV.

More details on Random Variables

def: Let $(\Omega, \mathcal{F}, P)$ be a probability space. A real-valued function X on Ω is a random variable if

$$\underbrace{\{\omega \in \Omega : X(\omega) \in B\}}_{\substack{\text{short hand is} \\ \{X \in B\} \text{ or } X^{-1}(B) \in \mathcal{F}}} \in \mathcal{F} \text{ for each } B \in \mathcal{B}.$$

Borel sets

Note: We are interested in the probability that X is a member of B for each Borel set B, i.e.

$$P(\{\omega \in \Omega : X(\omega) \in B\}) = P(\{X \in B\}) = P(X \in B)$$

But for this probability to exist, $\{X \in B\}$ must be an event.

Hence the defn of RV above.

Remark: Recall that a real-valued function f on Ω is $\mathcal{F}$-measurable iff $f^{-1}(B) \in \mathcal{F}$ for each $B \in \mathcal{B}$. Thus, random variables are just $\mathcal{F}$-measurable functions.

from #13:

* Measurability of a function depends only on the σ-algebra $\mathcal{F}$ and not on the prob. measure P.

One of the most important quantities associated with a RV is its probability distribution.

def: Let X be a RV on the probability space $(\Omega, \mathcal{F}, P)$. Then the probability distribution of X, denoted μ_X, is the set function on $\mathcal{B}$ defined by

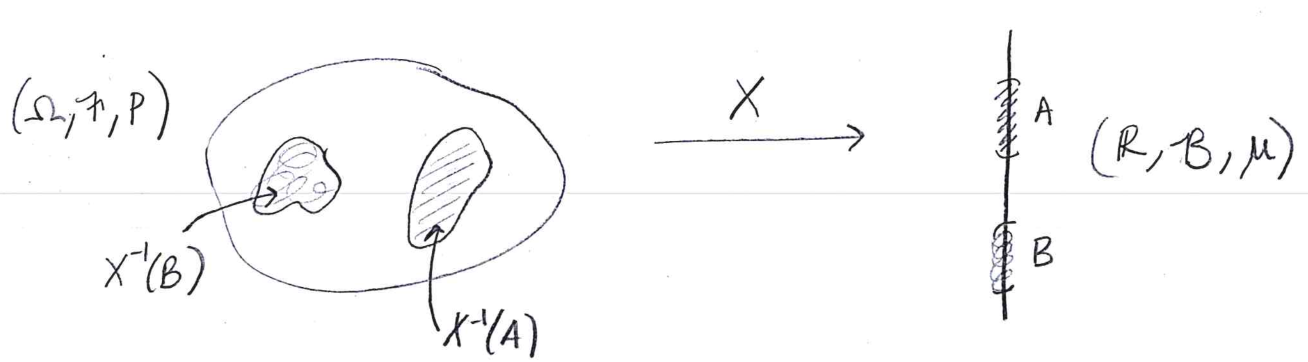
$$\mu_X(B) = P(X \in B) \quad \text{for } B \in \mathcal{B}.$$

e.g. PMF for discrete RV

e.g. PDF for continuous RV

def: Probability distribution function (aka CDF) of X:

$$F_X(x) = P(X \leq x) \quad \forall x \in \mathbb{R}$$



Pf: (i) If $X = 1_A$ for $A \in \mathcal{F}$, then for any subset $B \in \mathcal{F}$, $X^{-1}(B)$ must be one of the following:

only meas. sets → A, A^c, ϕ, or Ω. Hence $X^{-1}(B) \in \mathcal{F}$.

(See Rosenthal for remaining proofs!)

Suppose now that X is a RV and $f: \mathbb{R} \to \mathbb{R}$ is a function which is Borel-measurable, meaning that

$$f^{-1}(A) \in \mathcal{B} \text{ for any } A \in \mathcal{B}$$

where $\mathcal{B}$ = collection of Borel sets of $\mathbb{R}$.

[Equivalently, f is a RV corresponding to $\Omega = \mathbb{R}$ & $\mathcal{F} = \mathcal{B}$]

Define a new RV $f(X)$ by

$$\boxed{f(X)(\omega) = f(X(\omega))} \text{ for each } \omega \in \Omega.$$

↗ the composition of X with f

Note: this is well-defined since for $B \in \mathcal{B}$,

$$\{f(X) \in B\} = \{X \in f^{-1}(B)\} \in \mathcal{F}$$

Prop: If f is a continuous function, or a piecewise-continuous function, then f is Borel-measurable.

Pf: A basic result of point-set topology says: if f is continuous, then $f^{-1}(\mathcal{O})$ is an open subset of $\mathbb{R}$ whenever $\mathcal{O}$ is. In particular,

$f^{-1}((x,\infty))$ is open, so $f^{-1}((x,\infty)) \in \mathcal{B}$, so $f^{-1}((-\infty,x]) \in \mathcal{B}$.

If f is piecewise-cont., then we can write

$$f = f_1 1_{I_1} + f_2 1_{I_2} + \cdots + f_n 1_{I_n}$$

where f_j's are continuous and the $\{I_j\}$ are disjoint intervals. It follows from above & prev. prop. (3.1.5) that f is $\mathcal{B}$-measurable.

Example: $f(x) = x^k$ for $k \in \mathbb{N}$

f is Borel-measurable

Thus if X is a RV, then so is X^k $\forall k \in \mathbb{N}$.