STAT 706 Homework 1
SPRING 2019

Due on Thursday January 31 at the beginning of lecture.

1. Suppose that $\Omega = \{1, 2\}$, with $P(\emptyset) = 0$ and $P(\{1, 2\}) = 1$. Suppose $P(\{1\}) = \frac{1}{4}$. Prove that P is countably additive if and only if $P(\{2\}) = \frac{3}{4}$.

2. Suppose that $\Omega = \{1, 2, 3\}$ and $\mathcal{F}$ is the collection of all subsets of Ω.

   (a) List all the elements in $\mathcal{F}$.

   (b) Find necessary and sufficient conditions on the real numbers x, y, and z such that there exists a countably additive probability measure P on $\mathcal{F}$, with $x = P(\{1, 2\})$, $y = P(\{2, 3\})$, and $z = P(\{1, 3\})$.

3. Suppose that $\Omega = \mathbb{N}$ is the set of natural numbers, and P is defined for all $A \subseteq \Omega$ by $P(A) = 0$ if A is finite, and $P(A) = 1$ if A is infinite. Is P countably additive? Justify your answer.

4. Let $\Omega = \{1, 2, 3, 4\}$. Determine whether or not each of the following is a σ-algebra.

   (a) $\mathcal{F}_1 = \{\emptyset, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$

   (b) $\mathcal{F}_2 = \{\emptyset, \{3\}, \{4\}, \{1, 2\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$

   (c) $\mathcal{F}_3 = \{\emptyset, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3, 4\}\}$

5. Let
$$\mathcal{J} = \{\text{all intervals contained in } [0, 1]\}.$$
   Prove that
$$\mathcal{B}_0 = \{\text{all finite unions of elements of } \mathcal{J}\}$$
   is an algebra (or field) of subsets of $\Omega = [0, 1]$, meaning that it contains Ω and $\emptyset$, and is closed under the formation of complements and of finite unions and intersections.

6. Prove that $\mathcal{B}_0$, as defined in Problem 5 above, is not a σ-algebra.