STAT 706 Homework 3

SPRING 2019

Due on Thursday February 14 at the beginning of lecture.

1. Let $(\Omega_1, \mathcal{F}_1, P_1)$ be Lebesgue measure on $[0,1]$. Consider a second probability triple $(\Omega_2, \mathcal{F}_2, P_2)$ defined as follows: $\Omega_2 = \{1,2\}$, $\mathcal{F}_2$ consists of all subsets of Ω_2, and P_2 is defined by $P_2(\{1\}) = \frac{1}{3}$, $P_2(\{2\}) = \frac{2}{3}$, and additivity. Let $(\Omega, \mathcal{F}, P)$ be the *product measure* of $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$ (see definition on p.23 of Rosenthal).
   (a) Express each of Ω, $\mathcal{F}$, and P as explicitly as possible.
   (b) Find a set $A \in \mathcal{F}$ such that $P(A) = \frac{3}{4}$.
   (c) Give an example of a random variable X defined on $(\Omega_1, \mathcal{F}_1, P_1)$ (other than a uniform RV) . Verify that X is $\mathcal{F}_1$-measurable.
   (d) Give an example of a random variable Y defined on $(\Omega_2, \mathcal{F}_2, P_2)$. Verify that Y is $\mathcal{F}_2$-measurable.
2. Show that if events E and F are independent and $E \subset F$, then either $P(E) = 0$ or $P(F) = 1$.
3. Prove that if E and F are independent events, then so are E and F^C.
4. Suppose that $A_1, A_2, \ldots, A_n$ are mutually independent events.
   (a) Prove that $A_1 \cup A_2 \cup \cdots \cup A_{n-1}$ and A_n are independent events. [Hint: Use induction!]
   (b) Prove that $P\left(\bigcap_{k=1}^n A_k^c\right) = \prod_{k=1}^n P(A_k^c)$. [Hint: Use induction, part (a), and problem 3.]
5. Suppose that $\{A_n\} \nearrow A$. Let $f : \Omega \to \mathbb{R}$ be any function. Prove that

$$\lim_{n\to\infty} \inf_{\omega \in A_n} f(\omega) = \inf_{\omega \in A} f(\omega).$$