STAT 706 Homework 4
SPRING 2019

Due on Tuesday February 26 at the beginning of lecture. Note: R stands for exercises in Rosenthal's book.

1. (R 3.6.7)
   Let $(\Omega, \mathcal{F}, P)$ be the uniform distribution on $\Omega = \{1, 2, 3\}$. That is, $P(A) = |A|/|\Omega|$ for all $A \in \mathcal{F} = 2^\Omega$ (where $|A|$ is the cardinality of set A). Give an example of a sequence $A_1, A_2, \dots \in \mathcal{F}$ such that

   $$P\left(\liminf_{n\to\infty} A_n\right) < \liminf_{n\to\infty} P(A_n) < \limsup_{n\to\infty} P(A_n) < P\left(\limsup_{n\to\infty} A_n\right),$$

   i.e. such that all three inequalities are *strict*.

2. (R 3.6.9)
   Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $A_1, A_2, \dots, B_1, B_2, \dots$ be events in $\mathcal{F}$.

   (a) Prove that
   $$\left(\limsup_{n\to\infty} A_n\right) \cap \left(\limsup_{n\to\infty} B_n\right) \supseteq \limsup_{n\to\infty} (A_n \cap B_n).$$

   (b) Give an example where the above inclusion is strict, and another example where it holds with equality.

3. (R 3.6.11)
   Let $\{X_n\}_{n=1}^\infty$ be independent random variables with $X_n \sim \text{Uniform}(\{1, 2, \dots, n\})$ (see definition in Problem 1 above). Compute $P(X_n = 5\ i.o.)$, the probability that an infinite number of the X_n are equal to 5.

4. (R 3.6.16)
   Consider infinite, independent, fair coin tossing (see lecture 7 notes). Let H_n be the event that the n^{th} coin toss is heads. Determine the following probabilities:

   (a) $P(H_{n+1} \cap H_{n+2} \cap \dots \cap H_{n+9}\ i.o.)$.

   (b) $P(H_{n+1} \cap H_{n+2} \cap \dots \cap H_{2n}\ i.o.)$.

   (c) Prove that $P(H_{n+1} \cap H_{n+2} \cap \dots \cap H_{n+[\log_2 n]}\ i.o.)$ must equal either 0 or 1.

5. (R 3.6.13)
   Let $X_1, X_2, \dots$ be defined jointly on some probability space $(\Omega, \mathcal{F}, P)$, with $E[X_n] = 0$ and $E[(X_n)^2] = 1$ for all $n \in \mathbb{N}$. Prove that $P(X_n \geq n\ i.o.) = 0$.

6. (R 4.5.1)
Let $(\Omega, \mathcal{F}, P)$ be Lebesgue measure on $[0, 1]$ and set

$$X(\omega) = \begin{cases} 1, & 0 \le \omega < \frac{1}{4} \\ 2\omega^2, & \frac{1}{4} \le \omega < \frac{3}{4} \\ \omega^2, & \frac{3}{4} \le \omega \le 1. \end{cases}$$

Compute $P(X \in A)$ where

(a) $A = [0, 1]$.

(b) $A = [\frac{1}{2}, 1]$.