Further Classification of states (§4.3 con't)

f_i = Probability the process will ever re-enter state i, starting in state i

<u>def</u>: State i is <u>recurrent</u> if $\boxed{f_i = 1}$.

The process will re-enter state i infinitely often

<u>def</u>: State i is <u>transient</u> if $\boxed{f_i < 1}$.

Let X = # of times the process is in state i. (starting from i)

$$P(X=n) = f_i^{n-1} \underbrace{(1-f_i)}_{\text{"success" prob.}}, \quad n = 1,2,3,\ldots$$

geometric distribution with mean

$$E[X] = \frac{1}{1-f_i} < \infty$$

~~alternate version of geometric~~ $\cancel{P(X=n) = f_i^n (1-f_i), \quad n = 0,1,2,\ldots}$

(Prop. 4.1)
<u>Fact</u>: State i is <u>recurrent</u> iff $\sum_{n=1}^{\infty} P_{ii}^n = \infty$.

State i is <u>transient</u> iff $\sum_{n=1}^{\infty} P_{ii}^n < \infty$.

A transient state will only be visited a finite # of times.

Thus, for a finite state Markov chain, not all states can be transient. Why not?

More details ... on Prop 4.1

* state i recurrent iff starting in i, the expected # times process is in i is infinite.

Let $I_n = \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{if } X_n \neq i \end{cases}$

Then $\sum_{n=0}^{\infty} I_n$ = # time periods the process is in state i

$$E\left[\sum_{n=0}^{\infty} I_n \,\middle|\, X_0 = i\right] = \sum_{n=0}^{\infty} E[I_n \mid X_0 = i]$$ linearity of Exp. Value

$$= \sum_{n=0}^{\infty} P(X_n = i \mid X_0 = i) \cdot 1 + P(X_n \neq i \mid X_0 = i) \cdot 0$$

$$= \sum_{n=0}^{\infty} P_{ii}^{n}$$

So, i recurrent $\Leftrightarrow \sum_{n=0}^{\infty} P_{ii}^{n} = \infty$

i transient $\Leftrightarrow \sum P_{ii}^{n} < \infty$

Fact: Recurrence & Transience are class properties.

i.e. All states in 1 equivalence class are either all recurrent OR all transient.

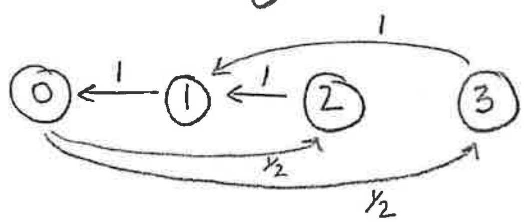
Note: If a Markov chain is finite and irreducible (meaning all states belong to the same class), then all states are recurrent.

*R MCsim3 Ex 1

Example: State space = $\{0, 1, 2, 3\}$

$$P = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Draw Diagram:



Q. Which states are transient?
Which states are recurrent?

All states are recurrent
$\Rightarrow$ irreducible MC

Check your work in R.

A few remarks:

- If i is recurrent & j is transient, then $i \not\leftrightarrow j$.
- Starting from a recurrent state, the MC cannot enter a transient state.

  (Converse IS possible)

- Once a MC enters a recurrent state, it can never leave that class.

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§4.4 For pairs of states $i \neq j$, let

$$f_{ij} = P(X_n = j \text{ for some } n > 0 \mid X_0 = i)$$

prob. that the MC, starting in state i, will ever make a transition into state j

If state i is recurrent and $i \leftrightarrow j$, then

$$f_{ij} = 1$$

Q. What is the average time (# of transitions needed) to come back to i? when starting in i

$$T_i = \min\{n \geq 1 : X_n = i\}$$ – first time returning to i

$$P(T_i = n) = f_i^{(n)}$$ – prob. first return takes n steps

see pg. 30 instead

mean time: $$E[T_i] = \sum_{n=1}^{\infty} n f_i^{(n)} = m_i$$

(same notation as book)

Long-run Proportions §4.4

Let i be a recurrent state of DTMC $\{X_n : n=0,1,\dots\}$.
Let m_i = expected # of transitions that MC takes, when starting in state i, to return to i.

$$m_i = E[N_i \mid X_0 = i] \quad \text{where} \quad N_i = \min\{n>0 : X_n = i\}$$

expected # transitions   # of transitions

def: The recurrent state i is

- positive recurrent if $m_i < \infty$ and
- null recurrent if $m_i = \infty$

Now suppose the MC is irreducible & recurrent. Then the long-run proportion of time the chain spends in state i is $\frac{1}{m_i}$.

Let π_i denote the long-run proportion of time spent in state i, then for any initial state,

$$\boxed{\pi_i = \frac{1}{m_i}} \quad \forall i \in S.$$

Proof: See Models §4.4 Prop 4.4