Recap of Discrete-time Markov Chains (DTMC)

$\{X_n : n = 0, 1, \dots\}$ is a DTMC on state space S.

Markov Property:

$P(X_{n+1} = j \mid X_n = i) = P_{ij}$ ← entry in $\mathbb{P}$ (transition matrix)

is independent of past history of MC, only depends on current state $X_n = i$.

Classification of States

$i \leftrightarrow j$ i & j communicate

If all states communicate, MC is irreducible

recurrent vs transient states

- positive recurrent if $m_i < \infty$
- null recurrent if $m_i = \infty$

where $m_i = E[N_i \mid X_0 = i]$, $N_i = \min\{n > 0 : X_n = i\}$

expected # transitions to return to i

# of transitions until MC moves to state i

Note: Quantity $\sum_{n=0}^{\infty} I_n$ counts # of times MC is in state i

NOT the same as m_i !!

(e.g. Say MC visits state i 10 times by time n & exp. return time is 50 time steps.)

- If i transient :

$$\sum_{n=1}^{\infty} P_{ii}^{n} < \infty$$ — only visit transient state a finite # of times

$P(N_i = \infty \mid X_0 = i) > 0$ and $m_i = \infty$

exp. return time

- If i is recurrent :

$$\sum_{n=1}^{\infty} P_{ii}^{n} = \infty$$ — return to recurrent state infinitely often

If $m_i < \infty$, then i is positive recurrent

If $m_i = \infty$, then i is null recurrent

Interpretation

A positive recurrent state – returns are relatively quick.

A null recurrent state – can take a very long time for MC to return to that state. (mean time $= \infty$)

Note : If a MC is finite, all recurrent states are positive recurrent.

(Can only have null recurrent state if S has ∞ # of states !)

Limiting Probabilities, Stationary Distribution §4.4 (con't)

Let $\{\pi_j, j \geq 0\}$ be the long-run proportion of transitions that come from state j

Then $\pi_i P_{ij}$ = long-run prop. of transitions from i to j

Thm: Consider an irreducible MC. If the chain is positive recurrent, then the long-run proportions are the unique solution of

$$(*) \quad \begin{cases} \pi_j = \sum_{i \in S} \pi_i P_{ij}, & j \geq 0 \\ \sum_{j \in S} \pi_j = 1. \end{cases}$$

If there is no solution to eqns above, then MC is either transient or null recurrent and all $\pi_j = 0$.

see EX →

Note: The long-run proportions $\pi_j, j \geq 0$ are called stationary probabilities.

$\pi = [\pi_1 \ \pi_2 \ \ldots \ \pi_j \ \ldots]$ — stationary distribution (for all $j \in S$)

More compactly written: $\boxed{\pi = \pi \mathbb{P}}$

vector matrix

Main Idea:

If $P(X_0 = j) = \pi_j$, then $P(X_n = j) = \pi_j \quad \forall n, j \geq 0$

start in state j

be in state j after n steps

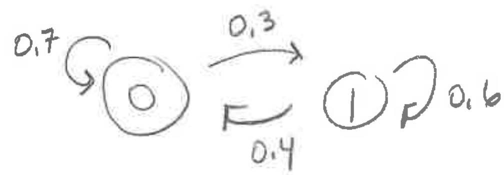
→ Example: Simple random walk on $\mathbb{Z}$

- symmetric RW is null recurrent $(p = \frac{1}{2})$
- asymmetric RW is transient $(p \neq \frac{1}{2})$

Limiting Probabilities

Consider the 2-state DTMC given by

$$\mathbb{P} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$



Note that

$$\mathbb{P}^4 = \begin{bmatrix} 0.575 & 0.425 \\ 0.564 & 0.433 \end{bmatrix} \quad \& \quad \mathbb{P}^8 = \begin{bmatrix} 0.572 & 0.428 \\ 0.570 & 0.430 \end{bmatrix}$$

very close & rows are almost identical

It seems that the long-run proportions

$\pi_0 = \frac{4}{7} \approx 0.571$ & $\pi_1 = \frac{3}{7} \approx 0.429$

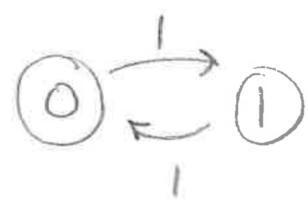
(computed in book § 4.4 Ex. 4.20) – solve eqns * to get these

may also be the limiting probabilities.

True in this case, but NOT true in general.

Counterexample – Periodic Chain

$$\mathbb{P} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$



This MC continually alternates b/t states 0 & 1

so $\pi_0 = \pi_1 = \frac{1}{2}$.

BUT $P_{00}^{n} = \begin{cases} 1 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$ so $P_{00}^{n} \to ??$ as $n \to \infty$

does NOT have a limiting value!!

* If a MC can only return to a state in a multiple of $d > 1$ steps ($d = 2$ in above example), it is called periodic; does NOT have limiting probabilities.

def: A MC that is not periodic is called aperiodic.
(all states have period 1.)

Thm: For an irreducible MC that is aperiodic, limiting probabilities always exist & do NOT depend on the initial state i.

$$\lim_{n \to \infty} P_{ij}^n \quad \text{exists}$$

Further, $\pi_j^* = \lim_{N \to \infty} \underbrace{\frac{\alpha_j(N)}{N}}$

limiting prob.

For large N, this is long-run proportion of time described before (π_j)

π_j^* satisfies eqns ⊛, unique soln.

where

$\alpha_j(N)$ = amount of time MC spends in state j during time $1, 2, \ldots, N$

def: State i has period d if

$P_{ii}^n = 0$ whenever n is NOT divisible by d,

and d is the largest integer with this property.

Periodicity = Class property: if $i \leftrightarrow j$, then i & j have same period.

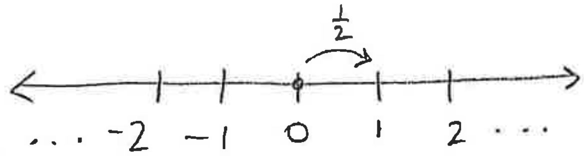
~~def: A aperiodic state is a state with period 1.~~

~~Periodicity = Class property: If $i \leftrightarrow j$, then i & j have the same period~~

Lec 11 3b

Example: Simple RW on $\mathbb{Z}$

- If you leave a state, can only come back in an even # of steps.
- Period of any state is 2. (Verify in R.)



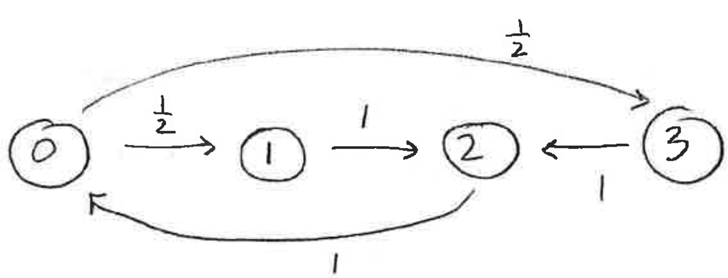
e.g. $0 \to 1 \to 0$ Return to 0 in 2 steps

did last Thurs

Example: MC on 4 states with

$$P = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

All states have period 3.
(Verify this in R.)



e.g. $0 \to 1 \to 2 \to 0$
OR
$0 \to 3 \to 2 \to 0$

Example 1: 3 state MC with transition matrix

$$\mathbb{P} = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

In the long run, what proportion of time is the process in each of the 3 states?

Soln: ① Solve $\pi = \pi \mathbb{P}$ (i.e. $\pi_j = \sum_{i=0}^{2} \pi_i P_{ij} \quad \forall j$)

$$\begin{cases} \pi_0 = 0.5\,\pi_0 + 0.3\,\pi_1 + 0.2\,\pi_2 \\ \pi_1 = 0.4\,\pi_0 + 0.4\,\pi_1 + 0.3\,\pi_2 \\ \pi_2 = \cdots \end{cases}$$

& $\pi_0 + \pi_1 + \pi_2 = 1$

OR ② put $\mathbb{P}$ into R & raise it to a large enough power to see columns converge to

$$P_i^n = [\pi_0 \quad \pi_1 \quad \pi_2]$$

Do in R

row i
(all same after some n)

Ans: $\pi_0 = \frac{21}{62} \approx 0.3387$, $\pi_1 = \frac{23}{62} \approx 0.3710$, $\pi_2 = \frac{18}{62} \approx 0.2903$

recurrent transient

Recall example from last week : $\{0,1\}, \{3\}, \{2\}$

no unique stationary distribution

SKIP Details

Mean Time Spent in Transient States §4.6

Recall: Once a MC enters a recurrent state, it can never leave that class

SKIP

$\Rightarrow$ Each recurrent class is a closed class

state i recurrent & $i \not\leftrightarrow j$ (does NOT communicate), then $P_{ij} = 0$.

In particular,

$$\left.\begin{array}{l} k \text{ recurrent} \\ j \text{ transient} \end{array}\right\} \Rightarrow P_{kj} = 0$$

Consider a finite state MC.

$T = \{1, 2, \dots, t\}$ denotes the set of transient states

$$\mathbb{P}_T = \begin{bmatrix} p_{11} & \cdots & p_{1t} \\ p_{21} & \cdots & p_{2t} \\ \vdots & & \vdots \\ p_{t1} & \cdots & p_{tt} \end{bmatrix}$$ — Matrix of 1-step transition probabilities from transient states to transient states

just say

(* sum of rows might be less than 1 ! why? o.w. T would be a closed class of states)

For $i, j \in T$,

s_{ij} = expected # of times MC is in state j given that it starts in state i

$$\Rightarrow \boxed{s_{ij} = \delta_{ij} + \sum_{k=1}^{t} p_{ik} s_{kj}}$$

Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

<u>In Matrix Notation</u> : $S = I + P_T S$

Identity matrix $(t \times t)$

where $S = \begin{bmatrix} s_{11} & \cdots & s_{1t} \\ \vdots & & \vdots \\ s_{t1} & \cdots & s_{tt} \end{bmatrix}$

Solving for S yields:

$$\boxed{S = (I - P_T)^{-1}}$$

Details

$S = I + P_T S$

$S - P_T S = I$

$S(I - P_T) = I$

$S = (I - P_T)^{-1}$

say

* Quantities s_{ij}, $i,j \in T$ can be obtained by inverting the matrix $I - P_T$

HW

Example (to do in R): Gambler's Ruin

$p = 0.4$

$N = 7$

$$P_T = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0.4 & & & & \\ 2 & 0.6 & 0 & 0.4 & & 0 & \\ 3 & & 0.6 & 0 & 0.4 & & \\ 4 & & & 0.6 & 0 & 0.4 & \\ 5 & & 0 & & 0.6 & 0 & 0.4 \\ 6 & & & & & 0.6 & 0 \end{array}$$

starting with \$3,

- compute the expected amount of time the gambler has \$5
- " " " gambler has \$2

i.e. Find $s_{3,5}$ and $s_{3,2}$

* use solve function in R

For $i, j \in T$,

f_{ij} = probability MC ever transitions into j given that it starts in i

$\Downarrow$

can be computed from s_{ij}'s

<u>Key Relation</u>

$$s_{ij} = \delta_{ij} + f_{ij} s_{jj}$$

$$\Rightarrow \boxed{f_{ij} = \frac{s_{ij} - \delta_{ij}}{s_{jj}}}$$