Lecture 12 Stat 753 2/27/20 1a

Clarification of Def. of Period of State i in DTMC

Period of state i:

$$k = \gcd \{ n > 0 : \underbrace{P(X_n = i \mid X_0 = i)}_{= P_{ii}^n} > 0 \}$$

So, suppose it is possible to return to state i in 2 or 4 steps. Then

$$\text{period of } i = \gcd \{2, 4\} = 2.$$

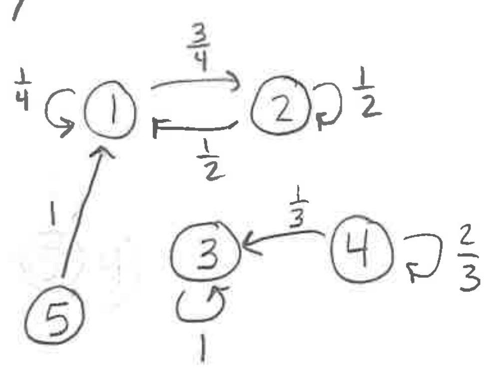
Example 1: Simple RW on $\mathbb{Z}$, either sym ($p = \frac{1}{2}$) or asym ($p \neq \frac{1}{2}$).

Possible to return to any state $i \in \mathbb{Z}$ in $2, 4, 6, 8, \ldots$ steps. The gcd of $\{2, 4, 6, \ldots\}$ is 2 so all states have period 2.

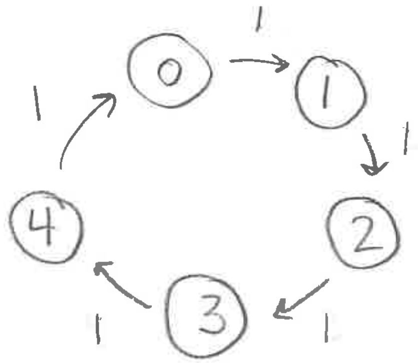
Example 2: DTMC defined by

SKIP HW example

$$\mathbb{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$



Example 3 : DTMC on RING graph



Every state has period 5.

Branching Processes

[ref: Models § 4.7 & other notes]

History: mid 19^{th} century, several aristocratic families in Victorian England realized that their family names could become extinct.

They asked the question:

How many male children (on average) must each generation of a family have in order for the family name to avoid extinction?

Galton & Watson (1874) "On the probability of extinction of families"

$\Longrightarrow$ Galton-Watson Branching Process Model

- Population starts with 1 individual in (time) generation 0 then build the family tree according to the rule:

$$\left\{ \begin{array}{l} \text{each individual in generation } n \\ \text{produces a random \# of offspring} \\ \text{in generation } n+1 \text{ (indep of other indivs.)} \end{array} \right. \quad \text{\& then dies}$$

- PMF is called the offspring distribution

$$p_i = P(\#\text{ of offspring} = i), \quad p_i < 1 \ \forall i$$

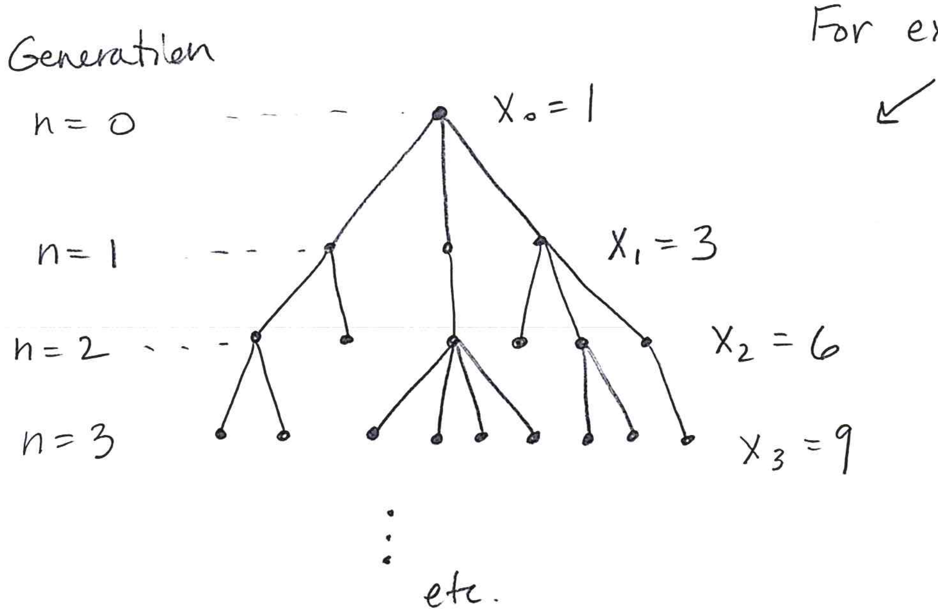
This process is a Markov chain:

X_n = size of the n^{th} generation

$\{X_n : n \geq 0\}$ is a MC where $n = 0, 1, 2, \ldots$ are generations

State space $= \mathbb{N}$

For example



Note: Size of the population at time n (X_n) depends on X_{n-1} but not on any previous generations. (Markov Property)

Note: If X_n reaches state 0, it stays there

$\Rightarrow$ 0 is an absorbing state (recurrent state)

and $P_{00} = 1$

- If $p_0 = P(X_n = 0) > 0$, then $\underbrace{P(X_{n+1} = 0 \mid X_n = k)}_{P_{k0}} > 0 \quad \forall k$

  $\Rightarrow$ all other states are transient

- If $p_0 > 0$, then the population will either die out OR its size will converge to ∞.
  (since any finite set of transient states will be visited finitely often)

- If $p_0 = 0$, then the population cannot decrease & increases each generation with probability at least $1-p_1$ (& therefore must tend to ∞).

Let $\mu = \sum_{i=0}^{\infty} i p_i$ = mean # of offspring of a single individual

$\sigma^2 = \sum_{i=0}^{\infty} (i-\mu)^2 p_i$ = variance of the # of offspring of a single indiv.

Let $X_0 = 1$. Then $E[X_0] = 1$ & $Var(X_0) = 0$.

It follows that

$$\boxed{\begin{aligned} E[X_n] &= \mu^n \\ Var(X_n) &= \begin{cases} \sigma^2 \mu^{n-1} \left(\frac{1-\mu^n}{1-\mu}\right) & \text{if } \mu \neq 1 \\ n\sigma^2 & \text{if } \mu = 1 \end{cases} \end{aligned}}$$

[See Models book for details, use the fact that

$$X_n = \sum_{j=1}^{X_{n-1}} Z_j \quad \text{where } Z_j = \text{\# of offspring of the } j^{th} \text{ indiv. in } (n-1)^{st} \text{ gen.}$$]

Q. What is the probability that the population will eventually die out?

Formulate this in terms of the stationary probability

$$\pi_0 = \lim_{n\to\infty} P(X_n = 0 \mid X_0 = 1) = ?$$

Main Result: (assumes $p_0 < 1$; (If $p_0 = 0$, then $\pi_0 = 0$))

$$\boxed{\mu \le 1 \Rightarrow \pi_0 = 1}$$ (population dies out)

$$\boxed{\mu > 1 \Rightarrow \pi_0 < 1}$$ (pop. persists)

and π_0 is the smallest positive # satisfying the equation

$$\pi_0 = \sum_{i=0}^{\infty} \pi_0^i p_i$$

(where $\mu = E[Z_i] \ \forall i$

$E[X_1] = \mu$

$E[X_2] = \mu E[X_1] = \mu^2$

$\vdots$

$E[X_n] = \mu E[X_{n-1}] = \mu \cdot \mu^{n-1} = \mu^n$

Can show that

$P(X_n \ge 1) \to 0$ when $\mu \le 1$

$\Rightarrow P(X_n = 0) \to 1$ " as $n \to \infty$