Lecture 16   Stat 753   10/19/17 (2)

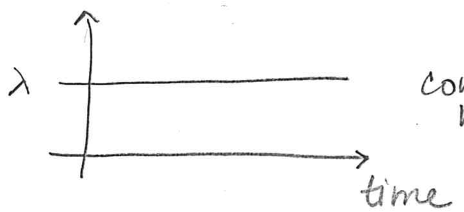
Nonhomogeneous Poisson Process

In many applications, the arrival rate changes over time (doesn't remain constant):

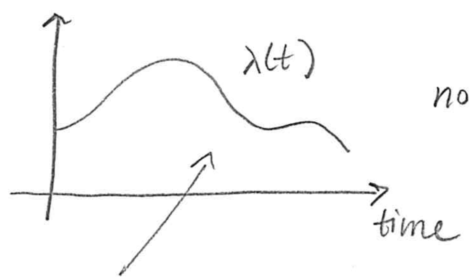
$$\lambda = \lambda(t)$$

function of time

Such a process is called a non homogeneous Poisson Process OR non stationary PP.



constant rate $\rightarrow$ homogeneous PP (HPP)



non constant rate $\rightarrow$ nonhomogeneous PP (NHPP)

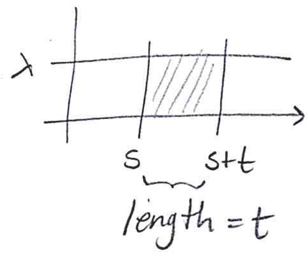
here, $\lambda = \lambda(t)$ is called the intensity function

If $\{N(t) : t \geq 0\}$ is a non homogeneous Poisson process with rate $\lambda(t)$, then

$N(s+t) - N(s)$ = # of events in the interval $(s, s+t]$, $\forall s \geq 0$

& has Poisson distn with parameter $\int_s^{s+t} \lambda(u)\,du$

Note: In HPP case, we're also integrating but it simplifies to



$$\int_s^{s+t} \lambda \, du = \lambda u \Big|_s^{s+t} = \lambda(s+t-s) = \lambda t$$

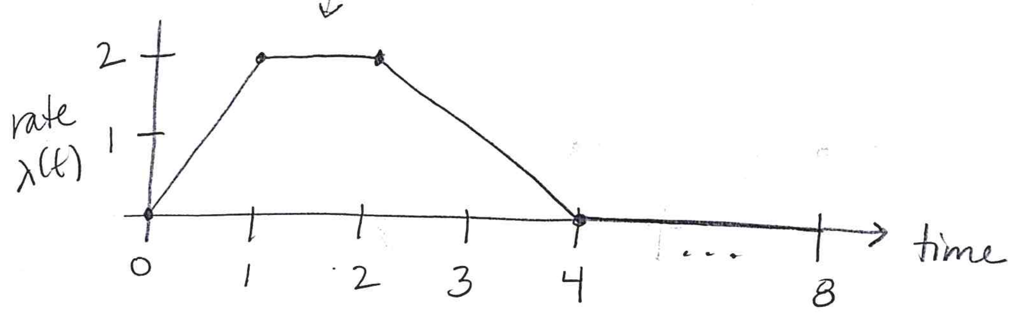
Note: In NHPP, the increments over disjoint intervals are independent RVs as in the HPP case.

Example: Demands on a first aid facility in a town occur according to a nonhomogeneous Poisson process with the following rate function

$$\lambda(t) = \begin{cases} 2t & \text{for } 0 \le t < 1 \\ 2 & \text{for } 1 \le t < 2 \\ 4-t & \text{for } 2 \le t < 4 \\ 0 & \text{for } 4 \le t < 8 \end{cases}$$

where t is measured in hours from the time the facility opens.

[Interpretation: Rush hour during 2nd hour of operation]



Q. What is the probability that 2 events (demands) occur in the 1^{st} 2 hours of operation & 2 in the 2^{nd} 2 hours?

Find $P(N(2)=2,\ N(4)-N(2)=2)$.

The time intervals $[0,2)$ & $[2,4)$ are disjoint

$\Rightarrow$ compute these two probabilities separately & multiply (Indep. events)

$$P(N(2)=2) = \frac{e^{-3}3^2}{2!}$$

Rate for 1^{st} 2 hrs $=$ Mean :
$$\int_0^2 \lambda(t)\,dt = \int_0^1 2t\,dt + \int_1^2 2\,dt$$
$$= 2\frac{t^2}{2}\Big|_0^1 + 2t\Big|_1^2$$
$$= 1 + (4-2) = 3$$

$$P(N(4)-N(2)=2) = \frac{e^{-2}2^2}{2!}$$

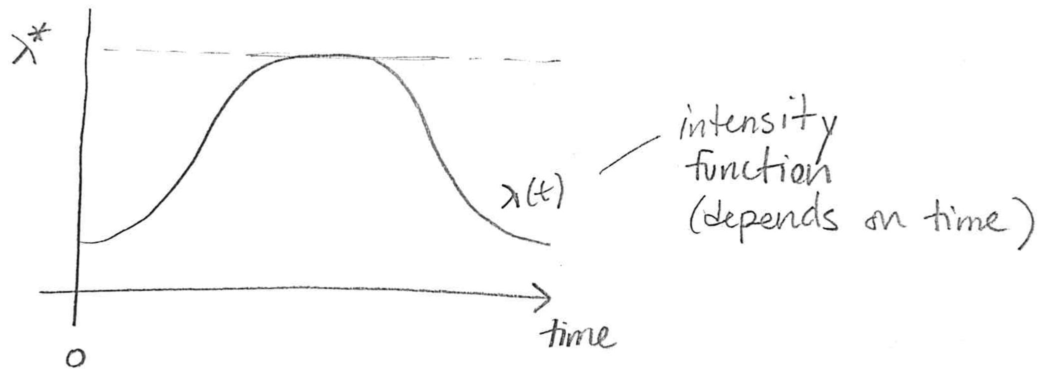
Rate for 2^{nd} 2 hrs $=$ mean :
$$\int_2^4 \lambda(t)\,dt = \int_2^4 (4-t)\,dt$$
$$= \left(4t-\frac{t^2}{2}\right)\Big|_2^4$$
$$= (16-8)-(8-2) = 2$$

Simulate both HPP & NHPP in R

using the "Poisson" package

Simulating a Nonhomogeneous Poisson Process

Common Approach : Based on "thinning" a homogeneous Poisson Process with rate $\lambda^* = \max(\lambda(t))$



Pseudocode :

1. Start with $t = 0$
2. Generate $U \sim U(0,1)$
3. Set $t \leftarrow t - \log(U)/\lambda^*$ (Also determine $\lambda^* = \max(\lambda(t))$)
4. Generate $V \sim U(0,1)$ independent of U
5. If $V \leq \lambda(t)/\lambda^*$, then keep event time t (otherwise discard it → "thin" the HPP this way)
6. Go back to step 2 (until some specified t_max has been reached)

<u>Note</u>: In step 3, $t - \underbrace{\log(U)/\lambda^*}$

current time

adds an exponential (λ^*) time increment, according to HPP rate $\lambda^* = \max(\lambda(t))$

<u>Note</u>: In step 5, $\lambda(t)/\lambda^*$ is the acceptance probability to keep the event at time t or not.

$\Rightarrow$ If $\lambda(t)$ is at the max value,

then $\lambda(t)/\lambda^* = 1$ & accept w/prob. 1

$\Rightarrow$ If $\lambda(t)$ is far below max value,

$\lambda(t)/\lambda^*$ is small, accept with small prob.

[Go to poisson-process-sim-nhpp.r

& poisson-process-sim.r

for examples to do in class!]

Note that there are other methods for simulating a NHPP

- Inverse CDF
- Order statistics

see R blog for 5 different methods

Compound Poisson Processes

def: A stochastic process $\{Z(t) : t \geq 0\}$ is a compound Poisson process if it can be represented as

$$\boxed{Z(t) = \sum_{i=1}^{N(t)} Y_i \quad , \quad t \geq 0}$$

where $\{N(t) : t \geq 0\}$ is a Poisson process, Y_i's are i.i.d. random variables $(i \geq 1)$ & independent of $\{N(t)\}$.

Examples

1. If $Y_i = 1 \ \forall i$, then $Z(t) = \sum_{i=1}^{N(t)} 1 = \underbrace{1 + \cdots + 1}_{N(t) \text{ times}} = N(t)$.

   (Usual Poisson Process)

2. $N(t)$ = # of customers that enter a store during the time interval $[0, t]$.

   Y_i = amount spent by customer i.

   $Z(t)$ = total amount of money customers spent in $[0, t]$.

3. $N(t)$ = # of buses that arrive at a sports event by time t.

   Y_i = # of fans on the i^{th} bus.

   $Z(t)$ = total # of fans that have arrived by time t.

Mean & Variance

Let $E[Y_1] = \mu$ and $Var(Y_1) = \sigma^2$.

Then $E[Z(t)] = \lambda t \mu$ and $Var(Z(t)) = \lambda t (\sigma^2 + \mu^2)$

λ: rate of Poisson process $\{N(t)\}$

Additive Property

$$\underbrace{Z_1(t) = \sum_{j=1}^{N_1(t)} Y_j^{(1)}, \quad Z_2(t) = \sum_{j=1}^{N_2(t)} Y_j^{(2)}, \quad \dots, \quad Z_n(t) = \sum_{j=1}^{N_n(t)} Y_j^{(n)}}_{n \text{ independent compound Poisson processes}}$$

$\{N_i(t)\}$ – independent, $i = 1, \dots, n$ (each with rate λ_i)

$\{Y_j^{(i)}\}$ – independent, $i = 1, \dots, n$, $j = 1, 2, \dots$

$\forall i \geq 1$, $Y_j^{(i)}$ are i.i.d. with CDF F_i

$\Downarrow$

$$Z(t) = Z_1(t) + Z_2(t) + \cdots + Z_n(t) \overset{D}{=} \sum_{j=1}^{N(t)} Y_j$$

$Z(t)$: Compound Poisson process

where $N(t) = N_1(t) + \cdots + N_n(t)$ is a Poisson process with rate $\lambda = \sum_{i}^{n} \lambda_i$

Y_j has CDF $F(x) = \sum_{i=1}^{n} p_i F_i(x)$, $p_i = \frac{\lambda_i}{\lambda}$

Generalizations of Poisson Process (con't)

Cox Process - a doubly stochastic Poisson process

$\lambda = \{\lambda(t) : t \geq 0\}$ ← rate λ is also a stochastic process!

Such processes are used to model spike trains (the sequence of action potentials generated by a neuron) & also in financial math to model prices of financial instruments in which credit risk is a significant factor.

(Possible Project Idea!)