Continuous-time Markov Chains

[Ref: Models ch. 6]

- We've already discussed one example: Poisson Process

  Let $N(t)$ = total # of arrivals by time t be the state of the process at time t.

  Then $\{N(t) : t \geq 0\}$ is a continuous-time MC

  $S = \{0, 1, 2, \ldots\}$

  Transitions only occur from state n to $n+1$ $(n \geq 0)$

  This type of process is also called a Pure Birth Process (or Yule Process).

- If we allow transitions from n to $n-1$ as well as from n to $n+1$, then we have a Birth & Death Process

(aka CTMC)

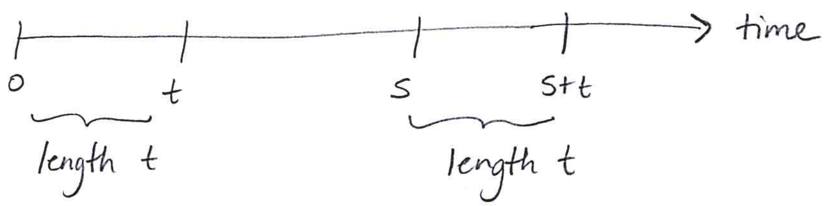
def: The process $\{X(t) : t \geq 0\}$ is a continuous-time MC if

$$P\big(X(t+s) = j \mid X(s) = i,\ X(u) = x(u) \text{ for } 0 \leq u < s\big)$$
$$= P\big(X(t+s) = j \mid X(s) = i\big)$$

Markov Property

$X(t)$ = state at time t (where t is continuous)

The continuous-time MC has stationary transition probabilities if $P(X(t+s)=j \mid X(s)=i)$ is independent of s.



- Let T_i = amount of time that the process stays in state i before transitioning into a different state.

  $T_i \sim$ exponential (v_i) ← this "waiting time" is exponentially distributed with rate v_i

  ⇒ average time until a transition is $1/v_i$

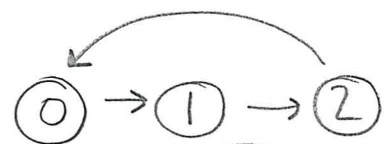
- When the process leaves state i, it transitions to state j with probability P_{ij}
  (Must satisfy $P_{ii}=0$ and $\sum_j P_{ij}=1$)

- The amount of time spent in state i & the next state visited must be independent. Why?

  (Info about how long the process has already been in state i would be relevant to the prediction of the next state → contradicts MP)

In other words, a continuous-time MC is a stochastic process that moves from state to state according to a discrete-time MC, but hangs out in each state for an exponential amount of time before transitioning.

"embedded Markov chain"



Example: Shoe Shine Shop

2 chairs (chair 1 & chair 2)

General sequential 2 stage process

A customer goes to chair 1 first → shoes are cleaned & polish applies

The customer moves to chair 2 next → polish is buffed

Service times at the 2 chairs are assumed to be independent exponential RVs with rates μ_1 & μ_2, resp.

Suppose that potential customers arrive according to a Poisson process with rate λ, and that a customer only enters the system if both chairs are empty.

# of customers in the system: 0 or 1

(But, if $\exists$ a customer in system, we need to know what chair he/she is in)

State Space of CTMC : 0, 1, 2

state	Interpretation
0	system is empty
1	a customer is in chair 1
2	a customer is in chair 2

Then transition probabilities

$$P_{01} = P_{12} = P_{20} = 1 \qquad (\text{\& all others are } 0)$$

And rates v_i (rate at which the process makes a transition when in state i)

$v_0 = \lambda$ ← customers arrive, Poisson process

$v_1 = \mu_1$ ← exponential (μ_1) service time in chair 1

$v_2 = \mu_2$ ← exponential (μ_2) " " chair 2

Instantaneous Transition Rates q_{ij}

For any pair of states i & j, let

$$\boxed{q_{ij} = v_i P_{ij}}$$

v_i is the rate defined above

P_{ij} is the probability that this transition is into state j

⇒ q_{ij} is the rate, when in state i, at which the process makes a transition into state j.

Then $\nu_i = \sum_j \nu_i P_{ij} = \sum_j q_{ij}$ and (since rows of P sum to 1)

$$P_{ij} = \frac{q_{ij}}{\nu_i} = \frac{q_{ij}}{\sum_j q_{ij}} .$$

Thus, specifying q_{ij} determines the parameters of the continuous-time Markov chain.

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Birth and Death Process

System whose state at any time = # of people in the system at that time.

Suppose that whenever there are i people in the system,

- new arrivals ("births") enter the system at an exponential rate λ_i
- people leave the system ("deaths") at an exponential rate μ_i

Such a system is a birth and death process.

$\{\lambda_i\}_{i=0}^{\infty}$ - birth rates

$\{\mu_i\}_{i=0}^{\infty}$ - death rates

A birth & death process is a CTMC with states $\{0, 1, \dots\}$, transitions from state i only go to state $i-1$ or $i+1$.

$$\begin{cases} i \to i+1 & \text{Birth} \\ i \to i-1 & \text{Death} \end{cases}$$

Transition Probabilities:

$$P_{01} = 1$$

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad i > 0$$

$$P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}, \quad i > 0$$

why? (see next page)

State transition rates:

$$\nu_0 = \lambda_0$$

$$\nu_i = \lambda_i + \mu_i, \quad i > 0$$

Redefine T_i s.t.

$T_{i,i+1}$ = time to go from i to $i+1$

$T_{i,i-1}$ = " " i to $i-1$

Then $T_{i,i+1} \sim \exp(\lambda_i)$

$T_{i,i-1} \sim \exp(\mu_i)$

$$\min(T_{i,i+1}, T_{i,i-1}) \sim \exp(\lambda_i + \mu_i)$$

Recall nice property of the exponential distn!

time to leave state i.