Then $\nu_i = \sum_j \nu_i P_{ij} = \sum_j q_{ij}$ and (since rows of P sum to 1)

$$P_{ij} = \frac{q_{ij}}{\nu_i} = \frac{q_{ij}}{\sum_j q_{ij}}.$$

Thus, specifying q_{ij} determines the parameters of the continuous-time Markov chain.

stopped

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Birth and Death Process

System whose state at any time = # of people in the system at that time.

Suppose that whenever there are i people in the system,

- new arrivals ("births") enter the system at an exponential rate λ_i
- people leave the system ("deaths") at an exponential rate μ_i

Such a system is a birth and death process.

$\{\lambda_i\}_{i=0}^{\infty}$ - birth rates

$\{\mu_i\}_{i=0}^{\infty}$ - death rates

A birth & death process is a CTMC with states $\{0, 1, \dots\}$, transitions from state i only go to state $i-1$ or $i+1$.

$$\begin{cases} i \to i+1 & \text{Birth} \\ i \to i-1 & \text{Death} \end{cases}$$

Transition Probabilities:

$P_{01} = 1$

$P_{i,i+1} = \dfrac{\lambda_i}{\lambda_i + \mu_i}$, $i > 0$

$P_{i,i-1} = \dfrac{\mu_i}{\lambda_i + \mu_i}$, $i > 0$

why? (see next page)

State transition rates:

$\nu_0 = \lambda_0$

$\nu_i = \lambda_i + \mu_i$, $i > 0$

Redefine T_i s.t.

$T_{i,i+1}$ = time to go from i to $i+1$

$T_{i,i-1}$ = " " i to $i-1$

Then $T_{i,i+1} \sim \exp(\lambda_i)$

$T_{i,i-1} \sim \exp(\mu_i)$

$$\min(T_{i,i+1}, T_{i,i-1}) \sim \exp(\lambda_i + \mu_i)$$

Recall nice property of the exponential distn!

time to leave state i.

$$P_{i,i+1} = P(T_{i,i+1} < T_{i,i-1}) = \frac{\lambda_i}{\lambda_i + \mu_i}$$

$$P_{i,i-1} = P(T_{i,i-1} < T_{i,i+1}) = \frac{\mu_i}{\lambda_i + \mu_i}$$

Also, state transition rate

$v_i = \lambda_i + \mu_i$ for $i > 0$.

Examples of Birth & Death Processes

- Poisson Process
- Pure Birth Process (e.g. Yule Process)
- Pure Death Process
- Linear Birth & Death Process (with & w/o immigration)
- Single server Exponential Queueing System (or Multiserver)

Example 1 : Poisson Process $\{N(t) : t \geq 0\}$

$N(t)$ = total # of arrivals by time t
(state at time t)

Poisson Process is a continuous-time MC
State space = $\{0, 1, 2, \dots\}$
Transitions : $i \to i+1$ only

Poisson Process is a special case of the Pure Birth Process :

- No departures / deaths
- Time between successive arrivals is exponential (λ), i.e. mean arrival time is $1/\lambda$

In other words,

$\{N(t) : t \geq 0\}$ is a Birth & Death Process s.t.

$\lambda_i = \lambda$ for all i

$\mu_i = 0$ for all i

$T_i \sim$ exponential (λ) for all i

$\nu_i = \lambda$ for all i

$$P_{ij} = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{o.w.} \end{cases}$$

Example 2: Pure Birth Process $\{X(t): t \geq 0\}$

A Birth & Death Process for which $\mu_i = 0 \ \forall i$.

When a transition occurs, the state is increased by 1 ($i \to i+1$ only).

Here, v_i may vary with i. ← More general than Poisson Process

$v_i = \lambda_i$ (birth rate)

$$P_{ij} = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{o.w.} \end{cases}$$ ← same as in PP

Example 3: Pure Death Process $\{X(t): t \geq 0\}$

A Birth & Death Process for which $\lambda_i = 0 \ \forall i$.

Transitions: $i \to i-1$ only

$v_i = \mu_i$ (death rate)

$$P_{ij} = \begin{cases} 1 & \text{if } j = i-1 \\ 0 & \text{o.w.} \end{cases}$$

[Yule Process]

Example 4 : Birth Process with Linear Birth Rate

Population whose members can give birth to new members but cannot die.

Assume that each individual acts independently of the others and takes an exponentially distributed amount of time with mean $1/\lambda$ to give birth

$\Rightarrow$ $\exp(\lambda)$

$X(t)$ = state at time t = population size at time t

Pure Birth process with

$\lambda_i = i\lambda$ for $i \geq 0$ (total birth rate)

$\nearrow$

i indivs in population
& each gives birth at
an exponential rate λ

Example 5 : Linear Birth & Death Process $\{X(t) : t \geq 0\}$

$$\begin{cases} \text{Each individual gives birth} \sim \text{exponential}(\lambda) \\ \text{Each individual dies} \sim \text{exponential}(\mu) \end{cases}$$

Suppose there are i individuals in the population.

Q. What is the time of the next birth ?
next death ?

Time of next birth:

minimum of i independent $\exp(\lambda)$ RVs

$T_{i,i+1} \sim \exp(i\lambda)$ ← linear birth rate

Time of next death:

min of i indep. $\exp(\mu)$ RVs

$T_{i,i-1} \sim \exp(i\mu)$ ← linear death rate

Thus, $\begin{cases} \lambda_i = i\lambda \\ \mu_i = i\mu \end{cases}$

Note that $\lambda_0 = 0 \Rightarrow$ state 0 is absorbing

Example 6: Linear Growth Model with Immigration $\{X(t): t \geq 0\}$

Suppose there are i individuals in the population.

- Each indiv. gives birth $\sim \exp(\lambda)$
- In addition, there is an exponential rate of increase θ in the population due to an external source, such as immigration.

$\Rightarrow$ Total birth rate: $\boxed{\lambda_i = i\lambda + \theta}$

- Each indivi dies $\sim exp(\mu)$

$\Rightarrow$ Total death rate: $\boxed{\mu_i = i\mu}$

- Such processes occur naturally in biology (biological reproduction, population growth & decay)

Example 7 : Queueing Systems

1) Single server exponential queueing system

2) Multiserver " " "

Single server case

Customers arrive at a single server service station according to a Poisson process with rate λ.

- Upon arrival, customer goes directly to service station if server is free

- If not, customer joins the "queue" i.e. waits in line

Successive service times are assumed to be exponential (μ) RVs, i.e. mean $= \frac{1}{\mu}$