- Each indivi. dies $\sim exp(\mu)$

$\Rightarrow$ Total death rate: $\boxed{\mu_i = i\mu}$

- Such processes occur naturally in biology (biological reproduction, population growth & decay)

Example 7: Queueing Systems

1) Single server exponential queueing system
2) Multiserver " " "

stopped

Single server case

Lec 20 1a.

Customers arrive at a single server service station according to a Poisson process with rate λ.

- Upon arrival, customer goes directly to service station if server is free
- If not, customer joins the "queue" i.e. waits in line

Successive service times are assumed to be exponential (μ) RVs, i.e. mean $= \frac{1}{\mu}$

Time of next birth:

minimum of i independent $\exp(\lambda)$ RVs

$T_{i,i+1} \sim \exp(i\lambda)$ ← linear birth rate

Time of next death:

min of i indep. $\exp(\mu)$ RVs

$T_{i,i-1} \sim \exp(i\mu)$ ← linear death rate

Thus, $\begin{cases} \lambda_i = i\lambda \\ \mu_i = i\mu \end{cases}$

Note that $\lambda_0 = 0 \Rightarrow$ state 0 is absorbing

Example 6: Linear Growth Model with Immigration $\{X(t): t \geq 0\}$

Suppose there are i individuals in the population.

- Each indiv. gives birth $\sim \exp(\lambda)$
- In addition, there is an exponential rate of increase θ in the population due to an external source, such as immigration.

$\Rightarrow$ Total birth rate: $\boxed{\lambda_i = i\lambda + \theta}$

Let $X(t)$ = # of customers in the system at time t.

$\{X(t) : t \geq 0\}$ is a birth & death process with

$$\begin{cases} \lambda_i = \lambda & \forall i \geq 0 \\ \mu_i = \mu & \forall i \geq 1 \end{cases}$$

Multiserver Case

- Customers arrive according to a Poisson process with rate λ.
- Successive service times are assumed to be indep. $\exp(\mu)$ RVs.
- Let there be s servers available, and
  $X(t)$ = # of customers in system at time t.

$\{X(t) : t \geq 0\}$ is a B & D process with

$$\begin{cases} \lambda_i = \lambda & \forall i \geq 0 \\ \mu_i = \begin{cases} i\mu & \text{if } 1 \leq i \leq s \\ s\mu & \text{if } i > s \end{cases} \end{cases}$$

[Example : Bank with s tellers]

- Suppose i customers in the system & $i \leq s$, then i servers are busy.

  → Each server works at rate μ so total departure rate $= i\mu$

- Suppose i customers in system and $i > s$, then all s servers are busy.

  → Total departure rate $= s\mu$

* These queueing systems are also known as

  M/M/1 queueing system ← single server case

  M/M/s queueing system ← multiserver

- First 'M' refers to Markovian arrival process (since it is a Poisson process)
- Second 'M' refers to the exponential service dist'n (& hence Markovian)
- '1' or 's' denotes single or multiserver case

There is also an <u>$M/M/\infty$ queueing system</u>.

Q. How to interpret this?

'∞' stands for infinitely many servers

$\Rightarrow$ there is <u>never a queue</u> so customers can immediately enter service

– assuming we start with a finite # of customers in the system

Q. Absurd model or useful??

Seems absurd, but actually useful for studying telephone traffic (for instance).

↳ To determine the # of phone lines (land lines) that are needed for UNR employees to call off-campus, we assume there are always enough ($M/M/\infty$ queue), find the stationary distribution & use this to estimate how many lines are needed s.t. 99% (or whatever accuracy you'd like!) of time there is enough capacity.

Transition Rate Matrix Q

In the $M/M/\infty$ queue,

$X(t)$ = # of customers in the system at time t

State space $S = \{0, 1, 2, \dots\}$

Matrix Q has entries $q_{ij} = v_i P_{ij}$ for $i \neq j$

jump rates

$$
\begin{array}{c|c}
 & \begin{matrix} 0 & 1 & 2 & 3 & 4 & \cdots \end{matrix} \\
Q = \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{matrix} &
\begin{bmatrix}
-\lambda & \lambda & & & & \\
\mu & -(\mu+\lambda) & \lambda & & & \\
 & 2\mu & -(2\mu+\lambda) & \lambda & & \\
 & & 3\mu & -(3\mu+\lambda) & \lambda & \\
 & & & & \ddots &
\end{bmatrix}
\end{array}
$$

Check:
Rows sum to 0 ✓

$$
\begin{tikzcd}
0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "2\mu"] \arrow[r, bend left, "\lambda"] & 3 \arrow[l, bend left, "3\mu"] \arrow[r, bend left, "\lambda"] & \cdots \arrow[l, bend left, "4\mu"]
\end{tikzcd}
$$

$i \to i+1$
Poisson Process w/rate λ (arrivals)

Exponential (μ) service times, i.e. $i \to i-1$ transitions occur at rate $i\mu$

In other words, the entries in Q are

$$
Q_{ij} = \begin{cases} q_{ij} = v_i P_{ij} & \text{if } i \neq j \\ -\sum_{j \neq i} q_{ij} = -v_i & \text{if } i = j \end{cases}
$$

so that the rows of Q sum to 0

$$
P_{ij} = \begin{cases} 1 & \text{if } j = i+1 \text{ or } j = i-1 \\ 0 & \text{o.w.} \end{cases}
$$

$q_{01} = v_0 P_{01} = \lambda \cdot 1 = \lambda$ $\qquad$ $q_{10} = v_1 P_{10} = \mu \cdot 1 = \mu$

In general $q_{i,i+1} = v_i P_{i,i+1} = \lambda \cdot 1 = \lambda$ $\qquad$ $q_{i,i-1} = v_i P_{i,i-1} = i\mu \cdot 1 = i\mu$

CTMC constructions (skipped in Lecture)

(1) $\{X(t) : t \geq 0\}$ is a CTMC if it can be constructed from an embedded MC $\{X_n\}$ with transition (aka discrete time) matrix P with the duration of a visit to state i having an exponential (ν_i) distribution.

(2) When $X(t)$ arrives at state i, generate i.i.d. exponential RVs $Y_j \sim \exp(q_{ij})$ where $q_{ij} = \nu_i P_{ij}$ for $j \neq i$.

Choose the next state to be $k = \arg\min_j Y_j$

and the time until the transition to be $\min_j Y_j$.
(i.e. duration spent in i)

Note that $\min_j Y_j \sim \exp(\nu_i)$

Example: A household consists of 4 individuals, some are healthy & some are infected with the flu. Suppose that infected individuals get well at rate μ, while each pair of indivs. has encounters at rate λ & the disease is transmitted if 1 is infected & the other is susceptible. (a) Formulate a continuous-time MC model for this system. (b) Simulate & estimate how long it might take until all members are infected for the case $\mu = 0$.

(a) $S = \{0, 1, 2, 3, 4\}$

$X(t) = \#$ of individuals infected at time t

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \mu & -(\mu+3\lambda) & 3\lambda & & \\ 0 & 2\mu & -(2\mu+4\lambda) & 4\lambda & \\ 0 & & 3\mu & -(3\mu+3\mu) & 3\lambda \\ 0 & & & 4\mu & -4\mu \end{bmatrix} \end{array}$$

infections

diag makes rows Σ to 0

recovery, linear

Assuming $X(0) \geq 1$

1 infected, 3 suscep. $\Rightarrow \binom{3}{1} = 3$ pairs of indivs w/ encounters at rate λ

$$\begin{tikzcd} 0 & 1 \arrow[l, "\mu"] \arrow[r, bend left, "3\lambda"] & 2 \arrow[l, bend left, "2\mu"] \arrow[r, bend left, "4\lambda"] & 3 \arrow[l, bend left, "3\mu"] \arrow[r, bend left, "3\lambda"] & 4 \arrow[l, bend left, "4\mu"] \end{tikzcd}$$

all healthy; no pairs ... $2 \cdot \binom{2}{1} = 4$ ... $\binom{3}{1} = 3$ pairs   all infected