Continuous-time Markov Chains (continued)

Transition Probability Function

Let $\boxed{P_{ij}(t) = P(X_{t+s} = j \mid X_s = i)}$ denote the probability that a process presently in state i will be in state j an amount of time t later.

$\Rightarrow$ These are the transition probabilities of the CTMC.

(In discrete-time MCs, transition probabilities P_{ij} is the probability of jumping from i to j in 1 step.

In continuous time, there's no first time $t > 0$ so we use the above definition of $P_{ij}(t)$ for each $t > 0$.)

We can explicitly determine $P_{ij}(t)$ in certain cases.

Example: $\{N(t) : t \geq 0\}$ is a Poisson process with rate λ. Let Y_n be a discrete-time MC with transition probabilities u_{ij}. Then $\{X_t = Y_{N(t)}\}$ is a CTMC that takes 1 jump according to u_{ij} at each arrival of $\{N(t)\}$.

In this example, $N(t)$ has a Poisson number of jumps with rate λt (= mean). Hence,

$$P_{ij}(t) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} u_{ij}^n$$

↑ Transition Probability for CTMC $\{X_t : t \geq 0\}$

↑ where u_{ij}^n is the n^{th} power of the discrete-time MC Y_n's transition prob. u_{ij}

Recall: Chapman-Kolmogorov Equation

$$\sum_k P_{ik}(s)\, P_{kj}(t) = P_{ij}(s+t)$$

Note: Transition Probabilities $P_{ij}(t)$ can be determined from their derivatives at 0:

$$q_{ij} = \lim_{h \to 0} \frac{P_{ij}(h)}{h} \quad \text{for } j \neq i$$

If this limit exists (it will for all cases we consider), we call q_{ij} the jump rate from i to j.

* In most cases, it is simpler to describe a CTMC by describing its transition RATES q_{ij} for $i \neq j$ rather than first figuring out $P_{ij}(t)$.

Recall: The rate at which the chain leaves state i is denoted by ν_i s.t.

$$\nu_i = \sum_{j \neq i} q_{ij}$$

def: The generator of the CTMC is the matrix Q such that

$$Q_{ij} = \begin{cases} q_{ij} & \text{if } i \neq j \\ -\nu_i & \text{if } i = j \end{cases}$$

← aka Instantaneous Transition Rate matrix

def: State i is absorbing if $\nu_i = 0$.

def: State i is stable if $0 < \nu_i < \infty$.

(If $\nu_i = \infty$, then the chain will leave i immediately so we will always assume $\nu_i < \infty$).

Kolmogorov's Equations

Q. How to obtain transition probabilities from the transition rates?

Theorem: Assume $\nu_i < \infty \ \forall i$ in the state space. Then the transition probabilities are differentiable (in t), and for any pair of states $i \neq j$

$$\frac{dP_{ij}(t)}{dt} = \sum_{k \neq i} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t) \qquad \text{Backward Equation}$$

$$\frac{dP_{ij}(t)}{dt} = \sum_{k \neq j} q_{kj} P_{ik}(t) - \nu_j P_{ij}(t) \qquad \text{Forward Equation}$$

In Matrix notation,

$$P'(t) = Q P(t) \qquad \text{Backward Eqn}$$

$$P'(t) = P(t) Q \qquad \text{Forward Eqn}$$

& Boundary Condition $P(0) = I$

* Rate matrix Q gives us everything we need to construct a CTMC, if we only had P we couldn't infer Q.

Example: Backward Eqns for Pure Birth Process $\quad P'_{ij}(t) = \lambda_i P_{i+1,j}(t) - \lambda_i P_{ij}(t)$

Stationary Distribution

Similar to discrete-time MC case, π is said to be the stationary distribution if

$$\boxed{\pi P = \pi.}$$

def: Let $\{X_t : t \geq 0\}$ be a CTMC with generator (rate matrix) Q. Then π is a stationary distribution if

$$\boxed{\pi Q = 0}$$

OR, equivalently, for each $j \in S$ the following equations hold:

$$\sum_{i \neq j} \pi(i) q_{ij} = \nu_i \pi(j)$$

def: We say π satisfies detailed balance for the CTMC if for each $i \neq j$

$$\boxed{\pi(i) q_{ij} = \pi(j) q_{ji}}$$

Thm: If π satisfies detailed balance, then π is a stationary distribution.

Gillespie's Stochastic Simulation Algorithm

Procedure for simulating sample paths of CTMCs.

Reactions: single instantaneous events changing at least 1 of the populations cause the system to change over time.

e.g. birth, death, movement, infection, etc.

Stochastic simulation algorithm (SSA):

samples the time τ to the next reaction & updates the system accordingly.

Each reaction R_j is characterized by 2 quantities:

- state change vector ν_j

  $\nu_j = (\nu_{1j}, \dots, \nu_{nj})$ where ν_{ij} is the change in # of indivs in pop. i caused by 1 reaction of type j

- propensity function $a_j(x)$

  i.e. prob. that reaction R_j will occur in $[t, t+dt]$ is $a_j(x)\,dt$

  $\frac{\text{rate } q_j}{\Sigma \text{ all rates}}$ — this is the correct probability