Renewal Theory

[Ref : Models Ch. 7]

Recall : A Poisson process is a counting process for which the times between successive events are i.i.d. exponential RVs.

- One generalization is to consider a counting process for which the times b/t successive events are i.i.d. with an arbitrary distribution.
- Such a counting process is called a renewal process.

def: Let $\{N(t) : t \geq 0\}$ be a counting process & let X_n be the time between the $(n-1)^{st}$ & n^{th} event, $n \geq 1$. If the sequence $\{X_1, X_2, \ldots\}$ is i.i.d., then the counting process $\{N(t) : t \geq 0\}$ is called a renewal process.

* When an event occurs, we say a renewal has taken place. Process starts itself over.

Example: Replacement of lightbulbs

Suppose we have an infinite supply of lightbulbs whose lifetimes are i.i.d.

We use a single lightbulb at a time, & when it fails we immediately replace it with a new one.

$\{N(t) : t \geq 0\}$ is a renewal process when

$N(t)$ = # of lightbulbs that have failed by time t.

Event = failure of lightbulb

Renewal Process → replace bulb immediately upon failure

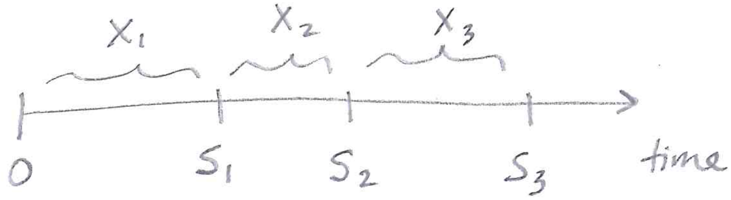
Inter-event Time = lifetime of a bulb

* Each event starts the process over with identical condition (aka "renewal")

$S_0 = 0$

$$S_n = \sum_{i=1}^{n} X_i \quad , n \geq 1$$

waiting time until the n^{th} event (renewal)



time of 1st renewal $= X_1$

time of 2nd renewal $= X_1 + X_2$

Let F denote the interarrival distribution ($X_i \sim F$).

Let $\mu = E[X_n]$ — mean time b/t successive renewals

Assume $P(X_i \geq 0) = 1$.
X_i's are nonnegative

- By the strong law of large numbers,

$$\frac{S_n}{n} \to \mu \quad \text{as } n \to \infty$$

with probability 1

(where S_n is the time of n^{th} renewal)

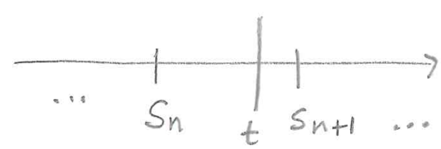
- $N(t) = \max\{n : S_n \leq t\}$

- $N(t)$ is finite for each t (with prob. 1)

- $N(\infty) = $ total # of renewals $= \infty$ (with prob. 1)
  $\left(\lim_{n\to\infty} N(t)\right)$

Relationship Between $N(t)$ and S_n

$$\boxed{N(t) \geq n \iff S_n \leq t}$$



# of renewals by time t $\geq n$

time of n^{th} renewal $\leq t$

$$P(N(t)=n) = P(N(t)\geq n) - P(N(t)\geq n+1)$$
$$= P(S_n \leq t) - P(S_{n+1} \leq t)$$

↗

allows us to compute the distribution of $N(t)$ from the distn of X_i's

Mean Value of $N(t)$

$$E[N(t)] = \sum_{n=1}^{\infty} n\, P(N(t)=n) = \sum_{n=1}^{\infty} P(N(t)\geq n) = \sum_{n=1}^{\infty} P(S_n \leq t)$$

↑ definition

↑ Alternative formula for integer-valued RVs (≥ 0)

def: The mean-value function OR renewal function of a renewal process is

$$\boxed{m(t) = \sum_{n=1}^{\infty} P(S_n \leq t)}$$

Notes: (1) $m(t)$ uniquely determines the renewal process, $\{N(t)\}$ & X_i's.

(2) $\{N(t)\}$ as a Poisson process

$\Rightarrow m(t) = \lambda t = t/\mu$ where $\mu = \frac{1}{\lambda} = E[X_i]$

Thus, (by (1) above) the Poisson process is the only renewal process having a linear mean-value function.

[$m(t) = \lambda t$ – linear function of t]

Renewal Equation

X_i – interarrival times, (continuous) PDF f & CDF F

$\{N(t) : t \geq 0\}$ – corresponding renewal process with renewal function $m(t) = E[N(t)]$

$$\Rightarrow \boxed{m(t) = F(t) + \int_0^t m(t-x) f(x)\, dx}$$

This is the renewal equation (often used to obtain $m(t)$)

Details: Condition on time of 1st renewal

$$m(t) = E[N(t)] = \int_0^\infty E[N(t) \mid X_1 = x] f(x)\, dx$$

2 cases: $\begin{cases} x \leq t & \Rightarrow N(t) \geq 1 \\ x > t & \Rightarrow N(t) = 0 \end{cases}$

Then, $E[N(t) \mid X_1 = x] = 1 + E[N(t-x)]$ if $x \le t$

$E[N(t) \mid X_1 = x] = 0$ if $x > t$

$$\Rightarrow \quad m(t) = \int_0^t E[N(t) \mid X_1 = x] f(x)\,dx$$

$$= \int_0^t \Big[1 + \underbrace{E[N(t-x)]}_{m(t-x) \text{ by def}}\Big] f(x)\,dx$$

$$= \int_0^t f(x)\,dx + \int_0^t m(t-x) f(x)\,dx$$

$$= F(t) + \int_0^t m(t-x) f(x)\,dx$$

↖ renewal equation

Example: $X_i \sim$ Uniform $(0,1)$

One instance in which the renewal eqn may be solved: interarrival distn $\sim U(0,1)$

$$m(t) = e^t - 1 \quad \text{for} \quad 0 \le t \le 1$$

(see book for details!)

Limit Theorems

We saw that, with probability 1, (aka "almost surely" a.s.)

$$N(t) \to \infty \quad \text{as } t \to \infty$$

Q. How fast does this happen?
(At what rate does $N(t)$ go to ∞?)

$$\lim_{t\to\infty} \frac{N(t)}{t} \; ?$$

- For a Poisson Process $\{N(t) : t \geq 0\}$,

$$E[N(t)] = m(t) = \lambda t = t/\mu \quad \text{where } \mu = \frac{1}{\lambda} = E[X_1]$$

X_i's $\sim \exp(\lambda)$

$$\frac{N(t)}{t} \to \lambda = \frac{1}{\mu} \quad \text{a.s.} \quad \text{as } t \to \infty$$

$$\left(\text{Note: } E\left[\frac{N(t)}{t}\right] = \frac{\lambda t}{t} = \lambda\right)$$

- This is true in general for a renewal process!

$$\boxed{\frac{N(t)}{t} \xrightarrow{a.s.} \frac{1}{\mu} \quad \text{as } t \to \infty}$$

where $\mu = E[X_1]$ and X_i's i.i.d.

Proof: see details in § 7.3 (Models Book)
Prop. 7.1

Details

Since $S_{N(t)}$ is time of last renewal prior to or at time t & $S_{N(t)+1}$ is time of 1^{st} renewal after time t, then

$$S_{N(t)} \le t < S_{N(t)+1}$$

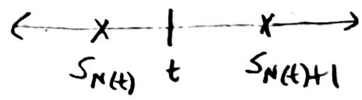
$$\Rightarrow \quad \frac{S_{N(t)}}{N(t)} \le \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}$$

Now since $\dfrac{S_{N(t)}}{N(t)} = \displaystyle\sum_{i=1}^{N(t)} \frac{X_i}{N(t)}$ is average of $N(t)$ i.i.d. RVs, it follows by SLLN that

$$\frac{S_{N(t)}}{N(t)} \to \mu \quad \text{as} \quad \underline{N(t)} \to \infty$$

But $N(t) \to \infty$ as $t \to \infty$, so we have

$$\frac{S_{N(t)}}{N(t)} \to \mu \quad \text{as} \quad \underline{\underline{t \to \infty}}.$$



Furthermore,

$$\frac{S_{N(t)+1}}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1} \cdot \frac{N(t)+1}{N(t)} \quad \Rightarrow \quad \frac{S_{N(t)+1}}{N(t)+1} \to \mu \text{ as } t \to \infty \text{ by same reasoning}$$

and $\dfrac{N(t)+1}{N(t)} \to 1$ as $t \to \infty$ $\Rightarrow$ $\dfrac{S_{N(t)+1}}{N(t)} \to \mu$.

Result follows by Squeeze Thm since $\dfrac{t}{N(t)}$ is b/t 2 RVs each converging to μ as $t \to \infty$.