Renewal Theory - Limit Theorems (continued) : §7.3 Models Book

Last time we showed that

$$\frac{N(t)}{t} \xrightarrow{a.s.} \frac{1}{\mu} \text{ as } t \to \infty, \quad \text{where } \mu = E[X_1]$$

and X_i's i.i.d. interevent times in the renewal process $\{N(t) : t \geq 0\}$.

- Interpret $\frac{1}{\mu}$ as the rate of the renewal process

- The average time between renewals is μ.

  $\Rightarrow$ average rate at which renewals occur is 1 per every μ time units.

Example 1 : Beverly has a radio that works on a single battery. As soon as the battery fails, she immediately replaces it with a new battery. If the lifetime of a battery (in hours) is uniformly distributed over $(30, 60)$, then at what rate does Beverly have to change batteries?

Ans: $N(t)$ = # of batteries that have failed by time t.

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} = \frac{1}{45}$$

(Since $\mu = E[X_1] = 45$

$X_1 \sim U(30,60)$ and $E[X_1] = \frac{1}{2}(30+60) = \frac{90}{2} = 45$)

Thus, in the long run, she'll have to replace a battery every 45 hours.

<u>Example 2</u>: Suppose that Beverly doesn't keep any extra batteries on hand, and so each time a failure occurs she must go & buy a new one. If the amount of time it takes her to get a new one is uniformly distributed over $(0,1)$, then what is the average rate that Bev. changes batteries?

<u>Ans</u>: In this case, mean time b/t renewals is

$$\mu = E[U_1] + E[U_2]$$

where $U_1 \sim U(30,60)$ and $U_2 \sim U(0,1)$.

$E[U_1] = 45$ ← as in Example 1

$E[U_2] = \frac{1}{2}(0+1) = \frac{1}{2}$

$\Rightarrow \mu = 45 + \frac{1}{2} = \frac{91}{2}$

Thus, the long run rate of changing batteries is $\frac{1}{\mu} = \frac{2}{91}$.

↙ i.e. 2 new batteries every 91 hrs.

Elementary Renewal Theorem

$$\frac{m(t)}{t} = \frac{E[N(t)]}{t} \to \frac{1}{\mu} \text{ as } t\to\infty.$$

$$\left(\text{when } \mu=\infty,\ \frac{m(t)}{t} \to 0 \text{ as } t\to\infty\right)$$

Note: This is not a simple consequence of the previous result $\left(\frac{N(t)}{t} \to \frac{1}{\mu} \text{ as } t\to\infty\right)$.

Seems reasonable to think that since the average renewal rate converges to $\frac{1}{\mu}$, then the expected average renewal rate also converges to $\frac{1}{\mu}$ ... but we must be careful here!

In general,
$X_n \to X$ a.s.
DOES NOT IMPLY
$E[X_n] \to E[X]$

See Example 7.8 in Models

In order to prove the Thm above, we need an identity known as **Wald's Equation**

To understand this, we need to introduce the concept of a **Stopping Time** def: The nonnegative integer-valued RV N is said to be a stopping time for a sequence of independent RVs $X_1, X_2, \ldots$ if the event $\{N=n\}$ is independent of $X_{n+1}, X_{n+2}, \ldots$ $\forall n=1,2,\ldots$

Main Idea: Suppose that the X_i's are observed in sequence. N denotes the # observed before stopping. Because the event that we stop after having observed $X_1, \ldots, X_n$ can only depend on these n values, & not on future values, it must be independent of these future values.

Examples of Stopping Times

1. Toss a coin: $X_i \sim \text{Bernoulli}(p=\frac{1}{2})$

   $N = \min\{n : X_1 + \cdots + X_n = 5\}$ is a stopping time.

   # of trials needed to get 5 heads

Q. Is $N = \min\{n : X_1 + \cdots + X_{n+1} = 5\}$ a stopping time?

No!

since here N depends on $n+1$
so $\{N=n\}$ depends on a future state

2. $X_i \sim \text{Bernoulli}(p)$.

$N = \min\{n : X_n = 1\}$ is a stopping time

$N = \min\{n : X_1 + \cdots + X_n = 25\}$ is a stopping time

$N = \min\{n : X_n = X_{n-1}\}$ is a stopping time

$N = \min\{n : X_n = \underline{\underline{X_{n+1}}}\}$ is <u>NOT</u> a stopping time

stopped

Wald's Equation

If $X_1, X_2, \ldots$ is a sequence of i.i.d. random variables with finite expectation $E[X]$, and if N is a stopping time for this sequence s.t. $E[N] < \infty$, then

$$\boxed{E\left[\sum_{n=1}^{N} X_n\right] = E[N]\,E[X]}$$

Pf: Details in book §7.3.