2. $X_i \sim$ Bernoulli(p).

$N = \min\{n : X_n = 1\}$ is a stopping time

$N = \min\{n : X_1 + \cdots + X_n = 25\}$ is a stopping time

$N = \min\{n : X_n = X_{n-1}\}$ is a stopping time

$N = \min\{n : X_n = X_{n+1}\}$ is NOT a stopping time

stopped

Wald's Equation

Lec 24 (1a)

If $X_1, X_2, \ldots$ is a sequence of i.i.d. random variables with finite expectation $E[X]$, and if N is a stopping time for this sequence s.t. $E[N] < \infty$, then

$$\boxed{E\left[\sum_{n=1}^{N} X_n\right] = E[N]\,E[X]}$$

Pf: Details in book §7.3.

Corollary of Wald's Equation

If $X_1, X_2, \ldots$ are the interarrival times of a renewal process, then

$$E[X_1 + X_2 + \cdots + X_{N(t)+1}] = E[X]\,E[N(t)+1]$$

Observe these X_i's 1 at a time & then stop at the 1st renewal after time t.

Then $N(t)+1$ is a stopping time for the seq. of interarrival times

That is,

$$E[S_{N(t)+1}] = \mu(m(t)+1).$$

Use this to prove the Elementary Renewal Thm.

Details

$$S_{N(t)+1} = \sum_{i=1}^{N(t)+1} X_i$$

$$\Rightarrow E[S_{N(t)+1}] = E\left[\sum_{i=1}^{N(t)+1} X_i\right] = \underbrace{E[X_i]}_{= \mu}\ \underbrace{E[N(t)+1]}_{= m(t)+1}$$

$$\Rightarrow E[S_{N(t)+1}] = \mu(m(t)+1)$$

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<u>Note</u>: X_i's and $N(t)$ are NOT independent!

$N(t) = n$ and $X_{n+1}, X_{n+2}, \ldots$ are dependent.

<u>BUT</u>

$\underbrace{N(t)+1 = n}$ and $X_{n+1}, X_{n+2}, \ldots$ are INDEPENDENT!

$N(t)+1$ is a stopping time for the sequence of interarrival times X_i's

$\Rightarrow$ By Wald's Equation,

$$E\left[\sum_{i=1}^{N(t)+1} X_i\right] = E[X_i]\, E[N(t)+1] = \mu\,(m(t)+1)$$

The Elementary Renewal Theorem follows from this relation.

(See details on p. 422 11th Ed Models)

$S_{N(t)+1}$ = 1st renewal time after time t

$$\Rightarrow S_{N(t)+1} = t + \underbrace{Y(t)}_{}$$

excess at time t
ie. the time from t until next renewal

$$\Rightarrow E[S_{N(t)+1}] = E[t + Y(t)]$$

$$\Rightarrow \mu(m(t)+1) = t + E[Y(t)]$$

$$\Rightarrow \frac{m(t)}{t} + \frac{1}{t} = \frac{1}{\mu} + \frac{E[Y(t)]}{t\mu} \qquad (*)$$

$$\Rightarrow \frac{m(t)}{t} \geq \frac{1}{\mu} - \frac{1}{t} \qquad \text{since } Y(t) \geq 0$$

$$\Rightarrow \lim_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu} \qquad \text{— bounded above (1)}$$

Suppose $\exists\ M < \infty$ s.t. $P(X_i < M) = 1$

X_i's are bounded

(see book for case of unbounded interarrival times)

$$\Rightarrow Y(t) < M$$

$$\Rightarrow E[Y(t)] < M$$

Then $$\frac{m(t)}{t} + \frac{1}{t} \leq \frac{1}{\mu} + \frac{M}{t\mu} \qquad (\text{from } (*))$$

$$\Rightarrow \lim_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu} \qquad \text{— bounded below (2)}$$

Hence, by (1) & (2), $\displaystyle\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}$. $\blacksquare$

Renewal Reward Process

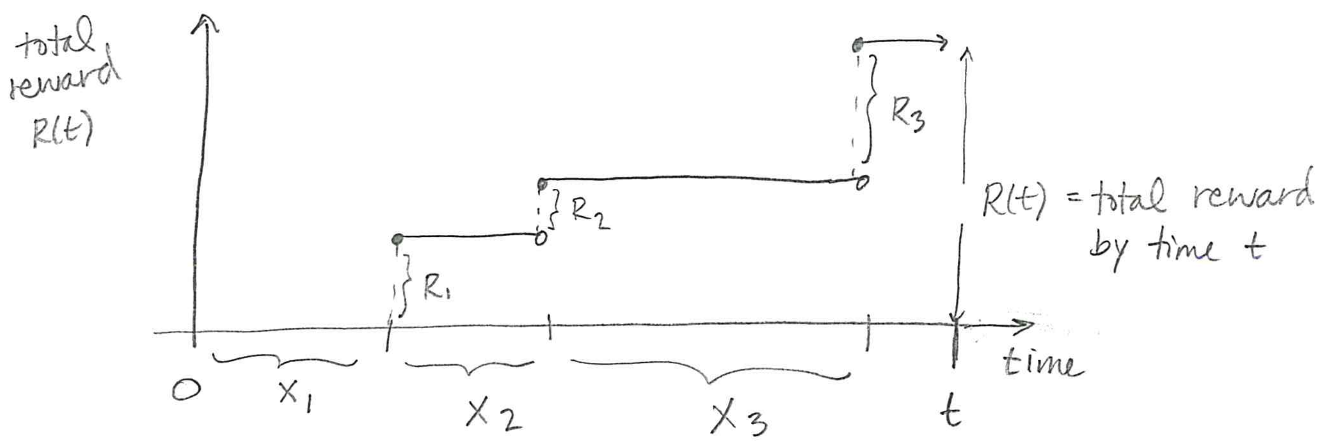
Suppose that each time a renewal occurs we receive a reward.

Let R_n = the reward earned at the time of the n^{th} renewal

Assume that the R_n's are i.i.d.

However, R_n may depend on X_n, the length of the n^{th} renewal.

$\{N(t) : t \geq 0\}$ is a renewal process with interarrival times $X_i \geq 0$.



Here, $N(t) = 3$ — 3 renewals in $[0,t]$

$$R(t) = \sum_{n=1}^{N(t)} R_n \quad = \text{total reward by time } t$$

($R(t)$ is also known as a continuous time random walk if X_i's have a continuous distn.)

Fact: If $\underset{E[R_n]}{E[R]} < \infty$ and $\underset{E[X_n]}{E[X]} < \infty$, then

1) $$\frac{R(t)}{t} \xrightarrow{a.s.} \frac{E[R]}{E[X]} \quad \text{as } t \to \infty$$

2) $$\frac{E[R(t)]}{t} \to \frac{E[R]}{E[X]}$$

Long-run average reward per unit time

$$= \frac{\text{Expected reward per cycle}}{\text{Expected length of cycle}}$$

(A cycle is completed every time a renewal occurs)

Example: Customers arrive at a single-server bank according to a Poisson process with rate λ. Customers only enter the bank if the server is free upon arrival, o.w. goes home. Assume the amount of time spent in bank is a RV with distribution G, then

(a) At what rate do customers enter the bank?

Skip ~~(b)~~ What proportion of potential customers actually enter the bank?

Let μ_G = mean service time.

Mean time b/t entering customers is

$$\mu = \mu_G + \frac{1}{\lambda} \quad \leftarrow \text{aka mean time of a cycle}$$

$$\Rightarrow \text{Rate customers enter bank} = \frac{1}{\mu} = \frac{\lambda}{1+\lambda\mu_G}.$$

Now suppose that the amounts that successive customers deposit in the bank are i.i.d. RV w/ distn H.

Then the rate at which deposits accumulate:

$$\lim_{t\to\infty} \frac{\text{Total deposits by time } t}{t} = ?$$

<u>Ans</u>:

$$\frac{E[\text{deposits during a cycle}]}{E[\text{time of a cycle}]} = \frac{\mu_H}{\mu_G + 1/\lambda}$$

where μ_H is the mean of distn H.