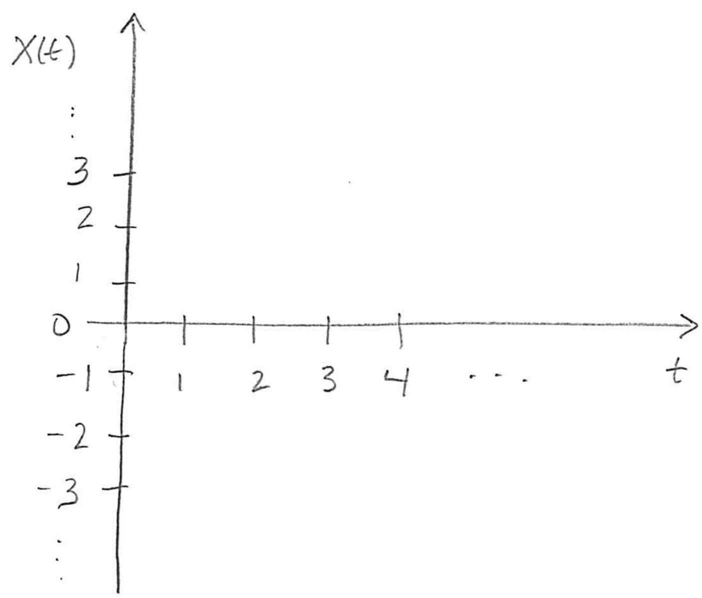
Brownian Motion

[Ref: Models Ch. 10]

Continuous-time continuous state stochastic process

Limit of symmetric random walk



Random Walk: Markov chain (discrete-time & space)

with $P_{i,i+1} = \frac{1}{2}$

$P_{i,i-1} = \frac{1}{2}$

for $i = 0, \pm 1, \pm 2, \ldots$ (or $i \in \mathbb{Z}$)

Each time step, the process moves up or down by 1 unit with probability $\frac{1}{2}$

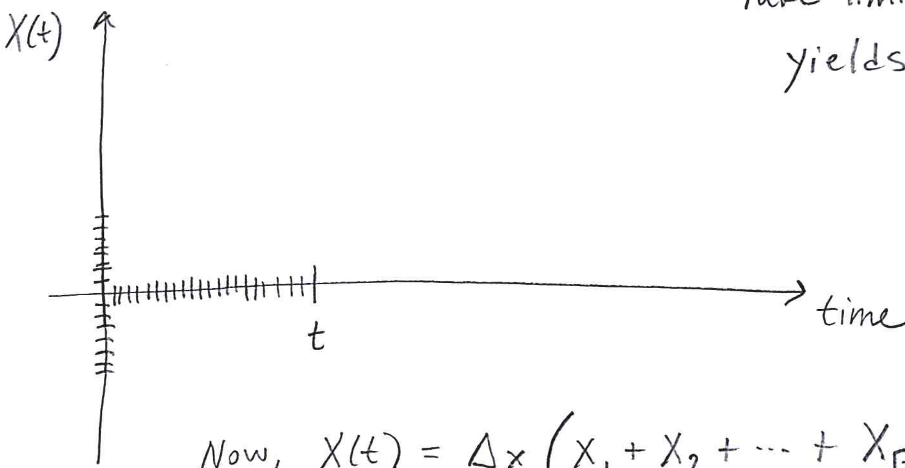
$\longrightarrow$ Rescale in time & space s.t.

time unit $= \Delta t$

space unit $= \Delta x$ — jump size

And let $\Delta t \to 0$, $\Delta x \to 0$. Apply CLT.

Take limit in right manner yields Brownian Motion.



Now, $X(t) = \Delta x \left(X_1 + X_2 + \cdots + X_{[t/\Delta t]} \right)$

where $X_i = \begin{cases} +1 & \text{if } i^{\text{th}} \text{ step of length } \Delta x \text{ is to right} \\ -1 & \text{if " " " is to left} \end{cases}$

$[t/\Delta t]$ = largest integer $\le t/\Delta t$

Also, X_i's assumed indep. with

$$P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$$

Then, $E[X_i] = 0$ and

$$Var(X_i) = E[X_i^2] = 1.$$

don't want $\Delta X = \Delta t$

o.w. mean & var $\to 0$

$$\Rightarrow \quad E[X(t)] = 0$$

$$Var(X(t)) = (\Delta x)^2 \left[\tfrac{t}{\Delta t} \right] \to \sigma^2 t \quad \text{as } \Delta t \to 0$$

(by letting $\Delta x = \sigma \sqrt{\Delta t}$ for some $\sigma > 0$)

Thus, $\{X(t)\}$ converges to a stochastic process on $[0, \infty)$ called Brownian Motion.

Example & Some Details :

Let $\sigma = 1$, $\Delta t = \frac{1}{100}$, $\Delta x = \sigma^2 \sqrt{\Delta t} = 1 \cdot \sqrt{\frac{1}{100}} = \frac{1}{10}$

Then $X(t) = \Delta x \left(X_1 + \cdots + X_{[t/\Delta t]} \right)$

$$= \frac{1}{10} \left(X_1 + \cdots + X_{[t \cdot 100]} \right)$$

So if $t = 2$, then

$$X(2) = \frac{1}{10} \left(X_1 + \cdots + X_{[200]} \right) = \frac{1}{10} \left(X_1 + \cdots + X_{200} \right) = \frac{1}{10} \sum_{i=1}^{200} X_i$$

$$E[X(2)] = \frac{1}{10} \sum E[X_i] = \frac{1}{10} \sum 0 = 0$$

$$Var(X(2)) = Var\left(\frac{1}{10} \sum_{i=1}^{200} X_i \right) = \left(\frac{1}{10} \right)^2 \sum_{i=1}^{200} Var(X_i)$$

$$= \frac{1}{100} \sum_{i=1}^{200} 1 = \frac{1}{100} \cdot 200 = 2$$

In general,

$$Var(X(t)) = Var\left(\Delta x \sum_{i=1}^{[t/\Delta t]} X_i \right) = (\Delta x)^2 \sum_{i=1}^{[t/\Delta t]} \underbrace{Var(X_i)}_{= 1 \ \forall i}$$

$$= (\Delta x)^2 \left[t/\Delta t \right]$$

$$\lim_{\Delta t \to 0} Var(X(t)) = \lim_{\Delta t \to 0} (\Delta x)^2 \left[t/\Delta t \right] \qquad \& \text{ since } \Delta x = \sigma \sqrt{\Delta t}$$

$$\leq \lim_{\Delta t \to 0} (\sigma \sqrt{\Delta t})^2 \cdot \frac{t}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \sigma^2 t = \sigma^2 t$$

def: A stochastic process $\{X(t): t \geq 0\}$ is said to be a Brownian Motion if

(i) $X(0) = 0$

(ii) $\{X(t): t \geq 0\}$ has independent increments:

For each $t_1 < t_2 < \cdots < t_n$, the RVs

$$\left\{ \begin{array}{l} X(t_1) \\ X(t_2) - X(t_1) \\ \vdots \\ X(t_n) - X(t_{n-1}) \end{array} \right.$$

are independent

(iii) $\{X(t): t \geq 0\}$ has stationary increments:

Distribution of $X(t+s) - X(t)$ does not depend on t, only the length of the interval s

(iv) For each $t > 0$, $X(t) \sim N(0, \sigma^2 t)$

$X(t)$ is Normally distributed with mean 0 & variance $\sigma^2 t$, $\sigma > 0$

Continuous sample paths, i.e. $X(t)$ is a continuous function of t (with probability 1)

- Brownian Motion is aka the Wiener Process

- One of the most useful stochastic processes in applied probability theory

- Originated in physics, named after Robert Brown who discovered it (1827):
    - the motion of a small particle (e.g. pollen grains in water) that is immersed in a liquid or gas
    - main idea: particle is continually subject to bombardment by the molecules of the surrounding media → displacement follows a normal distn "random motion"

* when $\sigma = 1$, the process is called standard Brownian Motion ← since for each $t > 0$ $X(t) \sim N(0, t)$

Often notated as $\{B(t) : t \geq 0\}$ or $\{B_t : t \geq 0\}$

Variations on Brownian Motion

- **Brownian Motion with drift**

$\{X(t): t \geq 0\}$ is a BM with drift coefficient μ and variance parameter σ^2 if

(i) $X(0) = 0$

(ii) $\{X(t): t \geq 0\}$ has stationary & independent increments

(iii) $X(t) \sim N(\underset{\text{mean}}{\mu t}, \underset{\text{var}}{\sigma^2 t})$

Normally distributed

Equivalently, let $\{B(t): t \geq 0\}$ be a standard Brownian Motion & then define

$$\boxed{X(t) = \sigma B(t) + \mu t}$$

- **Geometric Brownian Motion**

If $\{Y(t): t \geq 0\}$ is a BM with drift coeff. μ & var. σ^2, then the process $\{X(t): t \geq 0\}$ defined by $\boxed{X(t) = e^{Y(t)}}$ is called geometric BM.

Note: Geometric Brownian motion is useful for modeling stock prices over time when it seems that percentage changes are i.i.d.

(see Models §10.4 for more details on this)

R practice

brownian-motion-sim.r

OR

sde package / function BM