Lecture 26 | Stat 753 | 11/28/17 ①

Last Time: Brownian Motion

Continuous-time continuous state stochastic process

Limit of symmetric RW

Recall Definition: $\{X(t): t \geq 0\}$ is a Brownian Motion if

- $X(0) = 0$
- $\{X(t): t \geq 0\}$ has independent & stationary increments
- For each $t \geq 0$, $X(t) \sim N(0, \sigma^2 t)$

This is Brownian motion in 1 dimension.

Multivariate BM is similar:

$$X(t) = [X_1(t), X_2(t), \dots, X_k(t)]$$

$$X(t) \sim N_k(0, t\Sigma)$$

aka multivariate normal distn

K-dimensional normal distribution with mean 0 & covariance matrix $t\Sigma$

$$i,j \text{ element} = Cov(X_i(t), X_j(t)) = E[(X_i(t) - \mu_i)(X_j(t) - \mu_j)]$$

Standard Brownian Motion :

$\sigma = 1$ in 1D OR $\Sigma = I$ in k D

Sample paths of BM are continuous (with prob. 1)

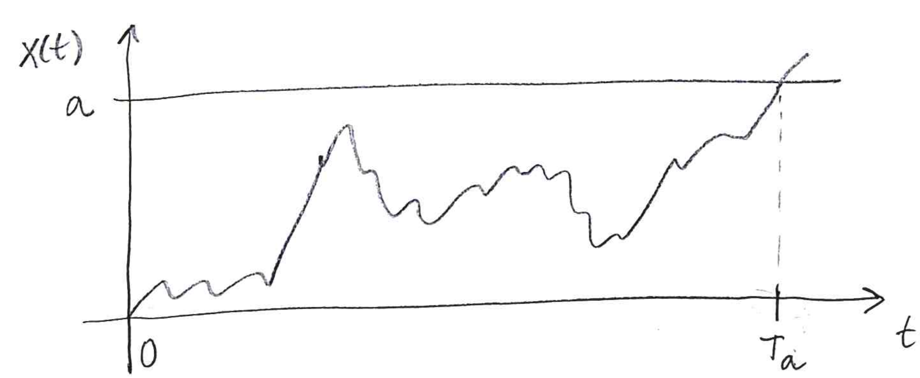
BUT nowhere differentiable !

Note : Brownian motion is a special case of Gaussian Processes.

Processes that satisfy the property:
All finite dimensional distributions are multivariate normal

Hitting Times

Let T_a denote the first time the Brownian motion (1D) $\{X(t) : t \geq 0\}$ process hits a. First suppose $a > 0$.



When $a > 0$, we can compute $P(T_a \le t)$ by considering $P(X(t) \ge a)$ and conditioning on whether or not $T_a \le t$.

This gives

$$P(X(t) \ge a) = P(X(t) \ge a \mid T_a \le t)\, P(T_a \le t) + P(X(t) \ge a \mid T_a > t)\, P(T_a > t) \quad (*)$$

[Q. where did this equation come from?]
Law of Total Probability

Now, $T_a \le t$ $\Rightarrow$ the process hits a at some point in $[0, t]$

$\Rightarrow$ by symmetry, it's just as likely to be above a or below a at time t.

$\Rightarrow$ $P(X(t) \ge a \mid T_a \le t) = \frac{1}{2}$

Note that the second term in $(*)$ above is 0.
Why? Process cannot exceed a if $T_a > t$.

Then $P(X(t) \geq a) = \frac{1}{2} P(T_a \leq t)$

$$\Rightarrow \quad P(T_a \leq t) = 2\, P(X(t) \geq a)$$

$$= \underbrace{2 \cdot \frac{1}{\sqrt{2\pi t}} \int_a^\infty e^{-x^2/2t}\, dx}$$

uses Normal PDF with mean $= 0$
& var $= t$ ($\sigma = 1$ standard BM)

$$\left[P(X(t) \geq a) = 1 - \underset{\substack{\uparrow \\ \text{Normal CDF}}}{P(X(t) \leq a)} \right]$$

Normal PDF: $X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(x-\mu)^2/2\sigma^2}$$

For the case $a < 0$,

the distribution of T_a is the same as that of T_{-a},

by symmetry.

Thus, $$\boxed{P(T_a \leq t) = \frac{2}{\sqrt{2\pi t}} \int_{|a|}^{\infty} e^{-x^2/2t}\, dx}$$

Another random variable of interest:

maximum value the process attains in the interval $[0,t]$.

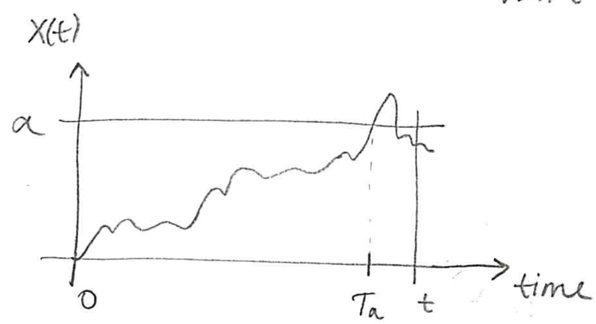
For $a>0$,

$$P\left(\max_{0\le s\le t} X(s) \geq a\right) = P(T_a \le t) \quad \text{by continuity}$$

$$= 2\,P(X(t)\geq a) \quad \text{from } (*)$$

$$= \frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-x^2/2t}\,dx$$

↑ distribution of max value RV



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def: A stochastic process $\{X(t): t \geq 0\}$ is called a Gaussian process if

$X(t_1), \dots, X(t_n)$ has a multivariate normal distribution for all $t_1, \dots, t_n$

"All finite dimensional distributions are multivar. normal"

Example 1: Let $\{X(t): t \geq 0\}$ be a Brownian motion process. Then since each of $X(t_1), X(t_2), \ldots, X(t_n)$ can be expressed as a linear combination of the independent normal RVs

$$X(t_1), \quad X(t_2) - X(t_1), \quad \ldots, \quad X(t_n) - X(t_{n-1}),$$

it follows that BM is a Gaussian process.

Example 2: Let $\{X(t): t \geq 0\}$ be a standard BM process, and consider the process values between 0 & 1 conditional on $X(1) = 0$.

That is, consider the conditional stochastic process

$$\{X(t), \; 0 \leq t \leq 1 \mid X(1) = 0\}.$$

The conditional distn of $X(t_1), \ldots, X(t_n)$ is multivar. normal so it follows that this process is a Gaussian process. Called the Brownian Bridge.

Name comes from fact that it is tied down both at 0 & 1

$$X(0) = 0 \quad \& \quad X(1) = 0$$

[Alternate def: $\{Z(t) = X(t) - t\,X(1), \; 0 \leq t \leq 1\}$]