Generalization of Brownian Motion:

Lévy Processes

def: A stochastic process $\{X(t): t \geq 0\}$ is called a Lévy Process if

- $X(0) = 0$
- $\{X(t): t \geq 0\}$ has stationary, independent increments

BUT $X(t)$ is not necessarily normally distributed!

Examples:

(1) Brownian Motion is a Lévy Process: $X(t) \sim N(0, t)$

(2) Poisson Process: $X(t) \sim \text{Pois}(\lambda t)$

(3) Compound Poisson Process

(4) Stable Process

Aka "jump-diffusion" processes

Note: BM is the only Lévy process with continuous sample paths.

def: A distribution is said to be stable if a linear combination of 2 indep. RVs with this distribution has the same distribution, up to location & scale parameters.

→ also studied by Lévy

i.e. Let X_1 & X_2 be indep with same distn as RV X.

Then X is stable if for any constants $a, b > 0$,

RV $aX_1 + bX_2 \stackrel{D}{=} cX + d$ for some constants $c, d > 0$.

"has the same distribution as"

Strictly stable if $d = 0$.

Special cases:

- Normal distribution
- Cauchy distribution ← interesting ("pathological") distn since exp. value & variance are undefined (∞) & all higher moments as well
- Lévy distribution

* In mathematical finance, Lévy processes are becoming extremely fashion able b/c they can describe the observed reality of financial markets in a more accurate way than models based on Brownian motion.

* Lévy processes are also used in
  - physics → study of turbulance, quantum field theory
  - engineering → study of networks, queves, dams
  - economics → continuous time series models
  - actuarial science → calculation of insurance & re-insurance risk

Stochastic Differential Equations (SDEs)

Overview: An SDE is a differential equation in which 1 or more of the terms is a stochastic process, resulting in a solution which is also a stochastic process.

Typically, SDEs contain a variable which represents "white noise" calculated as the (time) derivative of Brownian motion ... more on this later!

white noise — mathematical idealization of noise found in natural world

However, this comes at a cost: traditional calculus is no longer valid!

→ need to use stochastic calculus

Itô Stratonovich

$\underbrace{\hspace{6em}}_{\text{2 different versions}}$

Ordinary Differential Equation :

$$\begin{cases} \underbrace{\dfrac{dx(t)}{dt}}_{} \overset{\dot{x}(t)}{=} b(x(t)) & \text{for } t > 0 \\ x(0) = x_0 & \text{- initial condition} \end{cases}$$

where $b : \mathbb{R} \to \mathbb{R}$ is a smooth function of x & x_0 constant.

Solution is the trajectory x that solves the system above. Deterministic.

Stochastic Differential Equation

Idea: Modify an ODE to include the possibility of random effects disturbing the system

$$\begin{cases} \dot{X}(t) = b(X(t)) + B(X(t))\,\xi(t) & \text{for } t > 0 \\ X(0) = 0 \end{cases}$$

where $B : \mathbb{R} \to \mathbb{R}$ (in general we could have $B : \mathbb{R}^n \to \mathbb{M}^{n \times m}$ space of $n \times m$ matrices)

and $\xi(t)$ is "white noise"
$= dW(t)$