Stochastic Differential Equations (con't)

Recall from last time:

where $W(t)$ is Brownian motion (standard)

$$\text{SDE} \begin{cases} dX(t) = \underbrace{b(X(t))}_{\text{drift function}}\,dt + \underbrace{B(X(t))}_{\text{diffusion function}}\,dW(t) \\ X(0) = x_0 \end{cases}$$

This system is called a stochastic differential equation.

We say that X solves this SDE provided that

$$\boxed{X(t) = x_0 + \underbrace{\int_0^t b(X(s))\,ds}_{\text{Riemann integral}} + \underbrace{\int_0^t B(X(s))\,dW}_{\text{Stochastic (Itô) integral}}} \quad \text{for all times } t>0$$

Integral form

Stochastic (Itô) integral exists for the Wiener process (BM) & more general classes of processes [in the convergence in mean square sense]

Some Examples

- Standard Brownian Motion (aka Wiener process)

$$\text{SDE} \begin{cases} dX(t) = dW(t) \\ X(0) = 0 \end{cases} \iff X(t) = W(t)$$

General BM:

$$dX(t) = \sigma\, dW(t)$$

- Brownian motion with drift & scaling

$$\begin{cases} dX(t) = \mu\, dt + \sigma\, dW(t) \\ X(0) = 0 \end{cases} \iff X(t) = \mu t + \sigma W(t)$$

same as we discussed before

- Ornstein-Uhlenbeck process

$$\begin{cases} dX(t) = -\mu X(t)\, dt + \sigma\, dW(t) \\ X(0) = x_0 \end{cases}$$

$$\Updownarrow$$

$$X(t) = x_0 e^{-\mu t} + \sigma \int_0^t e^{-\mu(t-s)}\, dW(s).$$

- Double well potential (bimodal behavior, highly nonlinear)

$$dX(t) = \left(X(t) - X^3(t)\right) dt + dW(t)$$

- Geometric BM

$$\begin{cases} dX(t) = \mu X(t)\, dt + \sigma X(t)\, dW(t) \\ X(0) = x_0 \end{cases}$$

$$\Updownarrow$$

$$X(t) = x_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}$$

Numerical Methods for solving SDEs

- Numerical methods can be used to approximate the solution to SDEs
- Except in simple cases, generally not possible to obtain explicit solutions to SDEs
- Various methods:
  - Euler's method
  - Euler-Maruyama
  - Milstein
  - other Taylor approximations

Note: Let Z be a RV with a standard normal distribution: $Z \sim N(0,1)$.

Then $\sqrt{\Delta t}\, Z$ has a normal distribution with mean 0 & variance $(\sqrt{\Delta t})^2 = \Delta t$.

$$\Rightarrow \sqrt{\Delta t}\, Z \sim N(0, \Delta t)$$

Euler's method for solving the SDE on $[0,T]$

$$\begin{cases} dX(t) = b(X(t))\,dt + B(X(t))\,dW(t) \\ X(0) = x_0 \end{cases}$$

Assume initial condition x_0 is a fixed constant & $X(t)$ is the solution on $[0,T]$.

Partition the time interval $[0,T]$ into k subintervals of equal length: $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = T$

where $\Delta t = t_{i+1} - t_i = T/k$.

Then

$$t_{i+1} = t_i + \Delta t = i\,\Delta t$$

$$\Delta W_i = \Delta W(t_i) = W(t_i + \Delta t) - W(t_i)$$

<u>Euler's Method</u> $\Rightarrow$ $dX(t_i) \approx \Delta X(t_i) = X(t_{i+1}) - X(t_i)$

For each sample path, value of $X(t_{i+1})$ is approximated using only the value at the previous time step, $X(t_i)$.

Let X_i denote approximation to the full solution at time t_i.

Euler's Method is given by the recursive formula:

$$\boxed{X_{i+1} = X_i + b(X_i)\,\Delta t + B(X_i)\,\Delta W_i} \quad \text{\& } X_0 = x_0 \text{ initial condition}$$

for $i = 0, 1, \ldots, k-1$

To code this up, need to compute ΔW_i.

Since partition of $[0,T]$ has equal intervals

$$\Delta W_i \sim N(0, \Delta t) \quad \text{for all } i$$

Earlier we noted that if $Z \sim N(0,1)$, then

$$\sqrt{\Delta t}\, Z \sim N(0, \Delta t).$$

$$\Rightarrow \boxed{X_{i+1} = X_i + b(X_i)\,\Delta t + B(X_i)\,\sqrt{\Delta t}\, Z} \quad (*)$$

<u>Remark</u>: This formula also shows why sample paths corresponding to solns of SDEs are not differentiable.

Divide $(*)$ by Δt:

$$\frac{X_{i+1} - X_i}{\Delta t} = b(X_i) + B(X_i) \cdot \frac{Z}{\sqrt{\Delta t}}$$

Take limit as $\Delta t \to 0$:

LHS approaches $\frac{dX}{dt}$, but RHS does not exist!
$\left(1/\sqrt{\Delta t} \to \infty \right)$

* Can show that the numerical soln generated by Euler's method will have a dist'n that is close to the distn of the exact soln in the mean square sense on $[0,T]$.

i.e. $\lim_{n\to\infty} E\left[|X_n - X|^2\right] = 0$

R practice : sde-euler.r