Stochastic Processes

[ref: Models §2.9]

def: A stochastic process is a collection of random variables $\{X(t): t \in T\}$. For each $t \in T$, $X(t)$ is a RV.

- Index t is typically interpreted as time.
- $X(t)$ is the state of the process at time t.
- T is the index set of a process

  T countable (or finite) : discrete-time process

  T uncountable : continuous-time process
  (i.e. an interval)

- State space of $\{X(t): t \in T\}$ is the set of all possible values that $X(t)$ can take on. Discrete or Continuous.

Examples

cont. Space
1. $X(t)$ = total rainfall in day t ← discrete-time processes
   $T = \{0, 1, 2, \ldots, 365\}$

discrete Space
2. $X(t)$ = number of customers in a grocery store at time t (hours)
   $T = \{1, 2, \ldots, 24\}$

3. $X(t)$ = total number of customers that have entered a grocery store in the time interval $[0, t]$. Now $T = [0, \infty)$.

↗

Continuous-time process, discrete space

4. Brownian Motion

← random motion of particles suspended in a fluid

Continuous-time process, continuous state space ... get to this later in the semester!

5. Bernoulli Process

$X(t) \in \{0, 1\}$

$T = \{1, 2, 3, \ldots\}$

one of the simplest stochastic processes: sequence of coin flips (aka Bernoulli trials)

(discrete-time discrete space)

* Stochastic processes model the evolution of a random system over time.
(some physical process)

Example 1: (Gambling) Starting with \$K, bet \$1 on heads when tossing a coin.

Toss

H → win \$2

T → win \$0 (nothing)

Net winnings in 1 round of the game

\$1 if H

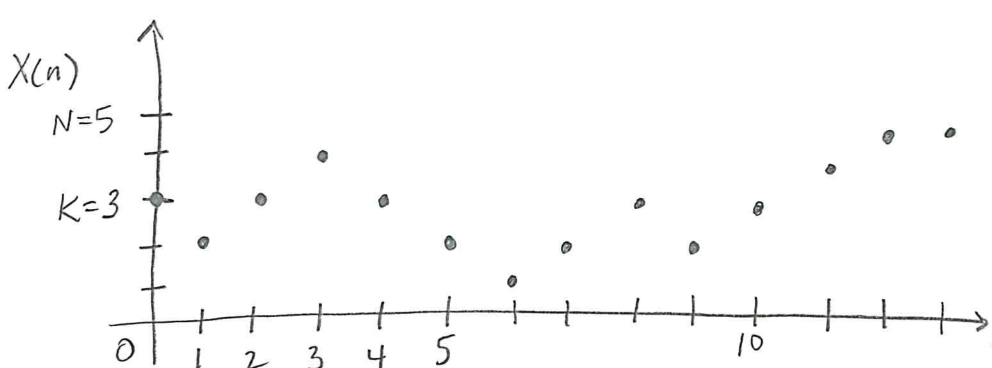
−\$1 if T

Keep playing until the player runs out of \$ or their total capital reaches \$N ($>K$).

$X(n)$ = the players capital at time n

state space $= \{0, 1, 2, \dots, N\}$

$\{X(n) : n = 0, 1, \dots, N\}$ – discrete-time stochastic process



$X(0) = 3$

$X(1) = 2$

$X(2) = 3$

$X(3) = 4$

$\vdots$

$X(12) = 5$

Sample "realization" of the process (aka sample path)

Example 2: $X_1, X_2, \ldots$ i.i.d. random variables

$\{X_n : n \geq 1\}$ stochastic process

trivial! No dependence among the X_i's

Example 3: $Y(n) = \sum_{i=1}^{n} X_i$

$Y_1 = X_1$

$Y_2 = X_1 + X_2$

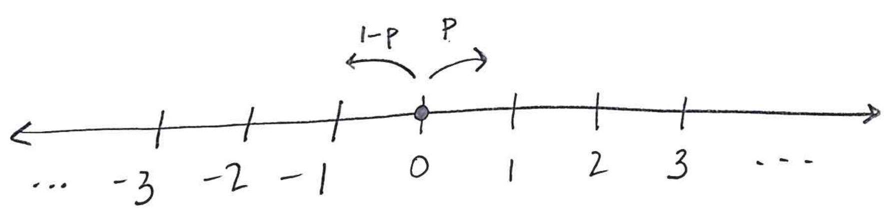
$Y_3 = X_1 + X_2 + X_3$

$\{Y(n) : n \geq 1\}$

Stochastic process with some dependence among the Y_i's

same as above

Example 4: Discrete-time Random Walk



$X_0 = 0$ initial state at time $n=0$

Given that $X_0 = 0$,

$P(X_1 = 1) = p$ ← move to right with probability p

$P(X_1 = -1) = 1-p$ ← move to left with prob. $1-p$

$\{X_n : n \in \mathbb{N}\}$ where state space $= \mathbb{Z}$ (integers)

<u>Sample Path</u> :

$X_0 = 0$

$X_1 = 1$

$X_2 = 2$

$X_3 = 1$

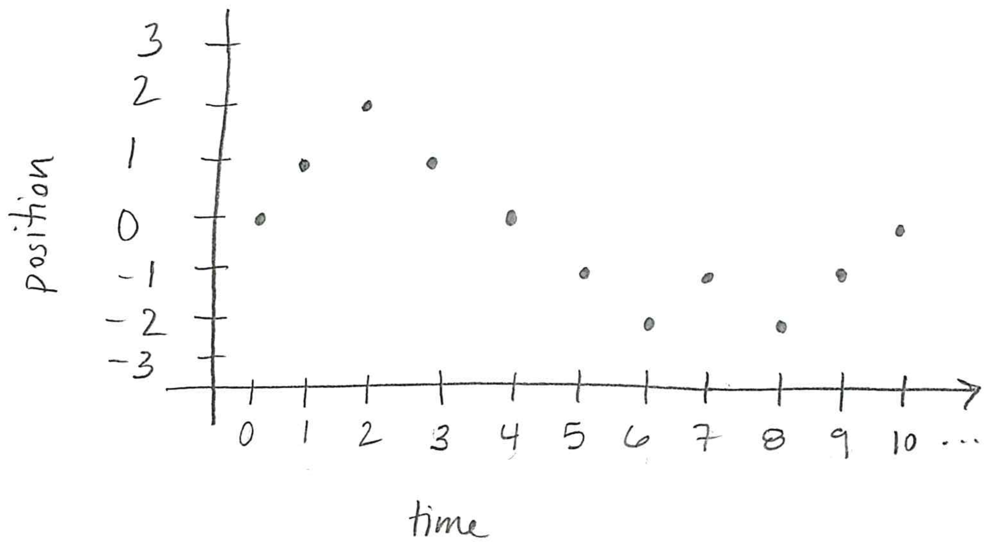
$X_4 = 0$

$X_5 = -1$

$X_6 = -2$

$\vdots$

Plot position as a function of time:



Compare to $Y_n = \sum_{i=0}^{n} X_i$ where $X_i \in \{+1, -1\}$ for $i \geq 1$

modified Bernoulli trial

$Y_0 = X_0 = 0$

$Y_1 = X_0 + X_1 = 0 + 1 = 1$

$Y_2 = X_0 + X_1 + X_2 = 0 + 1 + 1 = 2$

$Y_3 = \underbrace{X_0 + X_1 + X_2}_{} + X_3 = 2 - 1 = 1$

$\vdots$

$Y_n = Y_{n-1} + X_n$ (OR $Y_{n+1} = Y_n + X_{n+1}$)

n^{th} mod. Bernoulli trial

Leads into Markov Chains! Only depend on current state to determine next state.

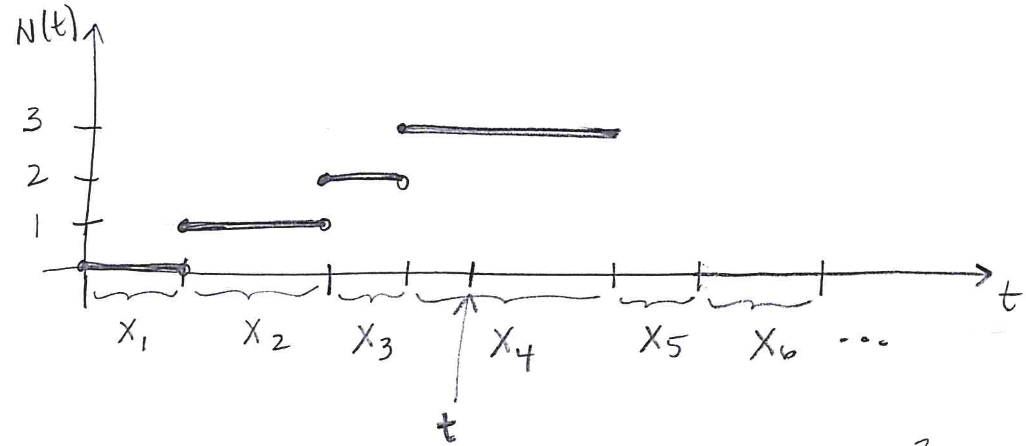
Example 5: Renewal Process (Poisson Process is a special case)

Assume X_i's are non-negative

$$N(t) = \max\{n : \sum_{i=1}^{n} X_i \le t\}, \quad t \ge 0$$

$\{N(t) : t \ge 0\}$ - continuous-time process

State space $= \mathbb{N} = \{0, 1, 2, \dots\}$



Here, $N(t) = 3$ because $\sum_{i=1}^{3} X_i < t$

and $\sum_{i=1}^{4} X_i > t$

Special case: X_i's i.i.d. exponential RVs,

then $\{N(t) : t \ge 0\}$ is a Poisson Process

* Renewal Process is more general since X_i's don't have to be exponential RVs (still i.i.d.)