Markov Chains

[ref: Models ch. 4]

Discrete-Time Markov Chain	vs.	Continuous-Time Markov Chain
index set → $T = \mathbb{N} = \{0, 1, 2, \dots\}$ (or finite subset)		$T = [0, \infty)$
State space $= \mathbb{N}$ ← for example		state space $= \mathbb{N}$

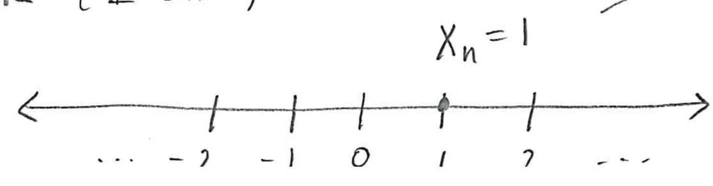
* Markov chain is a special type of stochastic process which has the Markov Property:

the conditional distribution of the future state given the current state & the past is independent of the past.

In other words,

the probability of any particular behavior of the process, when the current state is known, is not affected by additional knowledge of its past behavior.

e.g. Random Walk (1-dim) on $\mathbb{Z}$



→ $X_{n+1} = 0$ or 2

doesn't care about X_i for $i = 0, 1, \dots, n-1$

$\{X_n : n = 0, 1, \dots\}$ - discrete-time MC

<u>Transition Probability</u> from state i to state j at time n (1-step transition)

Discrete time case

$$P(X_{n+1} = j \mid X_n = i, \underbrace{X_{n-1} = i_{n-1}, \dots, X_0 = i_0}_{\text{past history}})$$

$$= P(\underbrace{X_{n+1} = j}_{\text{future state}} \mid \underbrace{X_n = i}_{\text{current state}})$$

$$= P_{ij}(n)$$

where $i_0, i_1, \dots, i_{n-1}, i, j, n \geq 0$

* Independent of <u>past history</u>!

Continuous time case

$$P(X_{t+s} = j \mid X_s = i, X_u = x(u), 0 \leq u < s)$$

all times u less than current time s

$$= P(X_{t+s} = j \mid X_s = i)$$

where $i, j, x(u) \geq 0$

$0 \leq u < s$

$$= P_{ij}(t)$$

In many examples, the transition probability does not depend on the time n:

If $\boxed{P(X_{n+1} = j \mid X_n = i) = P_{ij}} = P(X_1 = j \mid X_0 = i)$

No dependence on n!

then the Markov chain has <u>stationary transition probabilities</u>.

Arrange the P_{ij}'s into a Transition Probability Matrix P

i.e. matrix of 1-step transition probabilities

$$P = \begin{pmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ P_{20} & P_{21} & P_{22} & \cdots \\ \vdots & \vdots & \vdots & \\ P_{i0} & P_{i1} & P_{i2} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

$P_{ij} = P(X_{n+1} = j \mid X_n = i)$

= Prob. that the process makes the transition $i \to j$

$P_{ij} \geq 0$ & $\sum_{j=0}^{\infty} P_{ij} = 1$ — each row of P sums to 1

for $i = 0, 1, 2, \ldots$ since the process must make a transition into some state each time step

First Do Examples!

n-step transition probabilities

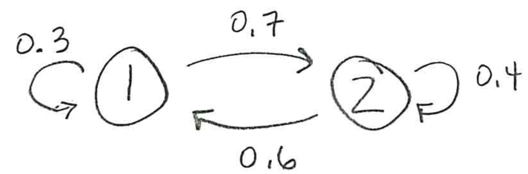
maybe Lec 9 ??

$P_{ij}^{n} = P(X_{n+k} = j \mid X_k = i)$, $n \geq 0$, $i, j \geq 0$

process moves from state i to j in n steps (transitions)

Q. Long term behavior of the MC ?

Example 1: 2-state Markov chain



move to next state in 1 step

$$P = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

rows sum to 1

current state

Example 2: Wright-Fisher model (genetics)

Model of random genetic drift
change in allele frequency of a gene over time

Consider a constant size population of $2N$ individuals:

2 types — A allele, a allele

If the current generation consists of

$$\begin{cases} i \text{ type A individuals} \\ 2N - i \text{ type a individuals} \end{cases}$$

then the next generation is determined by $2N$ Bernoulli trials with "success" probability

$$p_i = \frac{i}{2N} \quad (\text{for an A type indiv.})$$

and $\left(1 - p_i = 1 - \frac{i}{2N} = \frac{2N-i}{2N} \quad (\text{for an a type indiv.})\right)$

Define a MC $\{X_n : n = 0, 1, \dots\}$ such that

X_n = # of type A alleles in the n^{th} generation

i.e. Binomial RV w/parameters n & p_i

$S = \{0, 1, \dots, 2N\}$

Transition Probability:

$$P(X_{n+1} = j \mid X_n = i) = \underbrace{\binom{2N}{j} p_i^j (1-p_i)^{2N-j}}_{\text{Binomial PDF}} \quad \text{for } i, k = 0, 1, \dots, 2N$$

Note: 0 & 2N are called absorbing states

i.e. once the process enters 1 of these states (where all individuals are the same type), the process cannot leave that state.

If we modify the model to include mutation:

$$\mu_1 = P(A \to a) \quad \& \quad \mu_2 = P(a \to A)$$

then $$p_i = \frac{i}{2N}(1-\mu_1) + \left(1 - \frac{i}{2N}\right)\mu_2$$

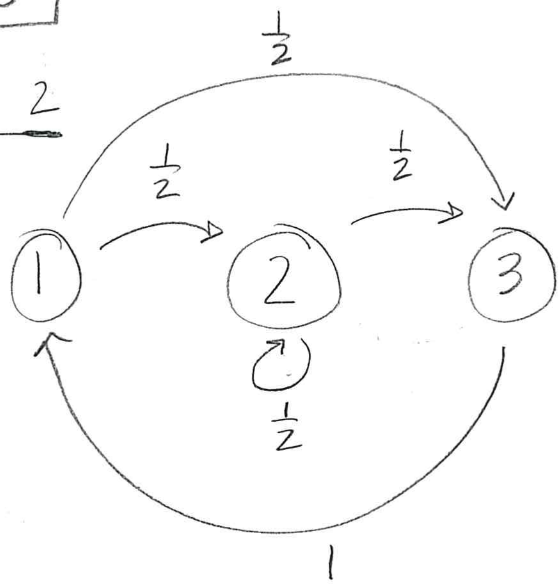
$\Rightarrow$ 0 & 2N are no longer absorbing states & long-term bevavior of MC stabilizes as $n \to \infty$.

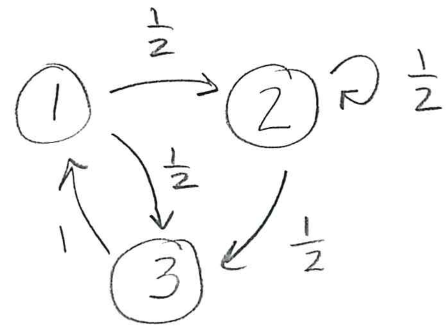
Lecture 8

R Practice

9/21/17 (4)
Stat 753

Example 2





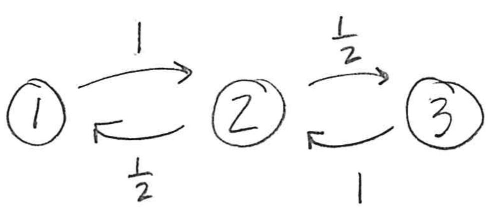
$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

states

Interpret as

Prob $\underset{\text{row}}{1} \rightarrow \underset{\text{col}}{2} = \frac{1}{2}$

Example 1



$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

states

Check:

* Rows of P sum to 1 ✓

Example 3

- modified version of Example 2:

2 is an absorbing state

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$