## Recap of Stochastic Processes thus far...

- Discrete-time Markov chains (discrete space)

  e.g. RW on $\mathbb{Z}$, branching process

- Continuous-time Markov chains (discrete space)

  e.g. Poisson process

  ↖ interevent times follow exponential dist'n, count events

  e.g. Renewal process

  ↖ interevent times follow arbitrary dist'n

- Continuous-time Markov Processes (Continuous space)

  e.g. Brownian Motion aka Wiener Process

  ↖ sample paths $X(t) \sim N(0, \sigma^2 t)$, continuous paths

  e.g. Lévy Process

  ↖ sample paths $X(t)$ are not necessarily normally distributed

- Stochastic Differential Equations

  Today!

## White Noise (§10.6)

First look at the case $m = n = 1$, $X_0 = 0$, $b \equiv 0$, $B \equiv I$.

Then

$$\begin{cases} \dot{X}(t) = 0 + 1\,\xi(t) \\ X(0) = 0 \end{cases}$$

has solution: $n = 1$ dim. Brownian motion, denote by $W(t)$

So $\dot{W}(t) = \xi(t)$

OR $\dfrac{dW(t)}{dt} = \xi(t)$ — time derivative of BM is white noise

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Now let $f$ be a function having continuous derivative in $[a,b]$. The stochastic integral is defined as

$$\int_a^b f(t)\,dW(t) = \lim_{\substack{n \to \infty \\ \max(t_i - t_{i-1}) \to 0}} \sum_{i=1}^{n} f(t_{i-1})\left[W(t_i) - W(t_{i-1})\right]$$

where $a = t_0 < t_1 < \cdots < t_n = b$ is a partition of $[a,b]$.

SKIP?

We can use the identity ← (integ. by parts formula applied to sums)

$$\sum_{i=1}^{n} f(t_{i-1})\left[W(t_i) - W(t_{i-1})\right] = f(b)W(b) - f(a)W(a) - \sum_{i=1}^{n} W(t_i)\left[f(t_i) - f(t_{i-1})\right]$$

Def:

to show that

$$\boxed{\int_a^b f(t)\,dW(t) = f(b)W(b) - f(a)W(a) - \int_a^b W(t)\,df(t)\,.}$$

Can show that:

$$E\left[\int_a^b f(t)\,dW(t)\right] = 0$$

$$Var\left[\int_a^b f(t)\,dW(t)\right] = \int_a^b f^2(t)\,dt$$

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* Strange form of chain rule in stochastic (Itô) calculus ...

Turns out that $\boxed{dW \approx \sqrt{dt}}$

X solves SDE $dX(t) = b(X(t)) + dW(t)$

$Y(t) = u(X(t))$ for $u: \mathbb{R} \to \mathbb{R}$ smooth function

Q. what SDE does Y(t) solve?

$$dY = u'\,dX \overset{?}{=} \underbrace{u'(b\,dt + dW)}_{\text{usual chain rule}} \quad \text{Wrong!}$$

$\left( ' = \frac{d}{dx} \right)$

$$\overset{\text{Yes}}{=} (u'b + \underbrace{\tfrac{1}{2}u''}_{\text{extra term!}})\,dt + u'\,dW$$

Recall Equation of an SDE:

$$\begin{cases} \dot{X}(t) = b(X(t)) + B(X(t))\,\xi(t) & \text{for } t > 0 \\ X(0) = 0 \end{cases}$$

Now write $\frac{d}{dt}$ instead of the dot:

$$\frac{dX(t)}{dt} = b(X(t)) + B(X(t))\,\frac{dW(t)}{dt}$$

where $W(t)$ is Standard Brownian Motion

and multiply by "dt":

$$(*) \quad \boxed{\begin{cases} dX(t) = b(X(t))\,dt + B(X(t))\,dW(t) \\ X(0) = 0 \end{cases}}$$

This is proper notation for an SDE (stochastic differential equation)

OR simply,

$$\begin{cases} dX = b\,dt + B\,dW \\ X(0) = 0 \end{cases}$$

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## Extra Details on Stochastic Integrals

– Ref: L. Allen's book ch. 8.7

**Example 1:** Simplest Itô stochastic integral

$$\int_a^b dW(t) = W(b) - W(a)$$

where $W(t)$ is standard Brownian motion

**Example 2:** For any well-defined random function $f(W(t))$

$$\left[ \text{s.t. } \int_a^b E\left(f^2(X(t))\right) dt < \infty \right],$$

$X$ stoch process

$$\int_a^b df(W(t)) = f(W(b)) - f(W(a))$$

Thm: Itô stochastic integral is a linear operator on the set of functions $f$ whose Itô stoch. integral exists.

**Example 3:** $$\int_0^t W(s)\, dW(s) = \frac{1}{2}\left[W^2(t) - t\right]$$

**Example 4:** $$\int_a^b W(t)\, dW(t) = \frac{1}{2}\left[W^2(b) - W^2(a)\right] - \frac{1}{2}(b-a)$$

* Itô integral NOT same as Rieman integral