STAT 753 Homework 1
SPRING 2020

Due on Thursday February 6 at the beginning of lecture.

1. If a fair coin is successively flipped, find the probability that a head first appears on the fifth trial.

2. Let the continuous random variables X and Y represent future lifetimes of Alice and Ben, respectively. The PDF of X is $f_X(x) = \alpha e^{-\alpha x}, x > 0$ and the PDF of Y is $f_Y(y) = \beta e^{-\beta y}, y > 0$, where α and β are positive constants. If the variables X and Y are independent, then what is the probability that Alice outlives Ben?

3. How large a random sample must be taken from a given distribution in order for the probability to be at least 0.99 that the sample mean will be within 2 standard deviations of the mean of the distribution? [Hint: Use Chebyshev's inequality!]

4. Suppose that X is a random variable for which the moment generating function (MGF) is
$$M(t) = \frac{1}{6}(4 + e^t + e^{-t}), \quad t \in \mathbb{R}.$$
Determine the probability $P(X \leq 0)$. [Hint: Try to guess the PDF of this distribution, prove that your guess is right, and use the PDF to find the probability].

5. Let $Z_1, Z_2, \ldots$ be a sequence of random variables, and suppose that for $n = 1, 2, \ldots$ the distribution of Z_n is as follows:
$$P(Z_n = n^2) = 1/n, \quad P(Z_n = 0) = 1 - 1/n.$$
Show that $E[Z_n] \to \infty$ and $Z_n \xrightarrow{p} 0$ (in probability) as $n \to \infty$.

6. Look up (see probability distribution chart) the moment generating function corresponding to the Poisson distribution with parameter λ, and use it to find the mean and the variance. Further, use the MGF to show that if $X_1, \ldots, X_n$ are independent and X_i has a Poisson distribution with parameter λ_i, then $X = X_1 + \cdots + X_n$ has a Poisson distribution with parameter $\lambda = \lambda_1 + \cdots + \lambda_n$.

7. **BONUS:** By using the MGF of Poisson distribution with parameter λ, establish the LLN and the CLT for this distribution. That is, show that $\bar{X}_n$ and Z_n converge in distribution to λ and $Z \sim$ N(0,1), respectively, where $\bar{X}_n$ is the sample mean of $X_1, \ldots, X_n$ and
$$Z_n = \frac{\bar{X}_n - \lambda}{\sqrt{\lambda}/\sqrt{n}}.$$