STAT 753 Homework 3
SPRING 2020

Due on Thursday February 20 at the beginning of lecture.

1. Let Y have a Weibull distribution $W(\alpha, \beta)$ with the survival function

$$S_Y(y) = Pr(Y \geq y) = \exp\left\{ -\left(\frac{y}{\alpha}\right)^{\beta} \right\}, \quad y \geq 0, \quad \alpha, \beta > 0.$$

Then Y has the same distribution as the random variable $\alpha(X)^{1/\beta}$, where X follows a standard exponential distribution (with parameter 1). Use the Inverse Transform Method to establish this fact via the following steps:

   (a) Find the CDF of Y, F_Y, and its inverse F_Y^{-1}.

   (b) Express Y as a function of a standard uniform random variable U using the Inverse Transform Method.

   (c) Express U as a function of X using the Inverse Transform Method.

   (d) Combine the results of parts (b) and (c).

2. Using the results of Problem 1, discuss how you would simulate a random variate from a Weibull distribution. Write an algorithm in R to generate a random sample of size n from this distribution. Illustrate by generating samples of size $n = 1000$ and $n = 10,000$ and by plotting the corresponding histograms. (Include both your R code and the 2 histogram figures).

3. Simulate a discrete-time random walk $\{X_n : n = 0, 1, 2, \dots\}$ on the integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ starting at 0. In particular, suppose that $X_0 = 0$ and

$$\begin{aligned} P(X_{n+1} = i + 1 \mid X_n = i) &= p \\ P(X_{n+1} = i - 1 \mid X_n = i) &= 1 - p \end{aligned}$$

for $n = 0, 1, 2, \dots$ and $i \in \mathbb{Z}$.

   (a) Generate 5 realizations of the process for $p = 0.5$ and plot sample paths on the same graph (position as a function of time for the first 100 time steps).

   (b) Generate 5 realizations of the process for $p = 0.8$ and plot sample paths on the same graph (separate graph from part (a), use 100 time steps).

   (c) Discuss the differences between the two processes.

4. Find and describe an example of an interesting real-world process that one could model with a Markov chain (MC). Categorize the MC as discrete or continuous in time and space, and spell out the index set (time T) and state space (S). Further, describe (mathematically) the transition probability matrix $\mathbf{P}$.